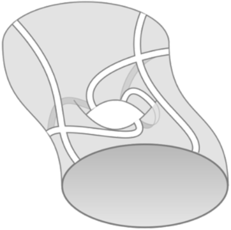
WESTFÄLISCHE
WILHELMS-UNIVERSITÄT
MÜNSTER

COLOURED TENSORS

THE WARD-TAKAHASHI IDENTITY



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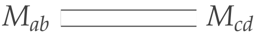
Belgrade, 26.August

- Matrix models techniques: in NC-QFT [Grosse, Wulkenhaar] $\varphi^{\star 4}$ (in Moyal-$\mathbb{R}^4_\Theta$)

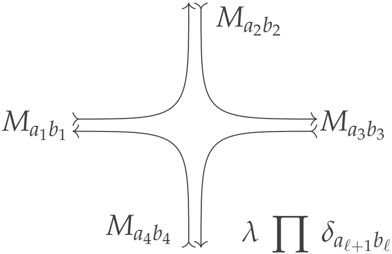
$$S = \int_{\mathbb{R}^4} \mathrm{d}x \left(\frac{1}{2} \varphi(-\Delta + \mu^2 + \Omega^2 ||2\Theta^{-1}x||^2)\varphi + \frac{\lambda}{4}\varphi^{\star 4} \right)(x)$$

$$Z[J] = \frac{\int \mathrm{e}^{\mathrm{Tr}(JM) - \mathrm{Tr}(EM^2) - \frac{\lambda}{4}\mathrm{Tr}(M^4)} \mathcal{D}M}{\int \mathrm{e}^{-\mathrm{Tr}(EM^2) - \frac{\lambda}{4}\mathrm{Tr}(M^4)} \mathcal{D}M}$$

- Feynman diagrams are ribbon graphs



$$E^{-1}_{ab;cd}$$



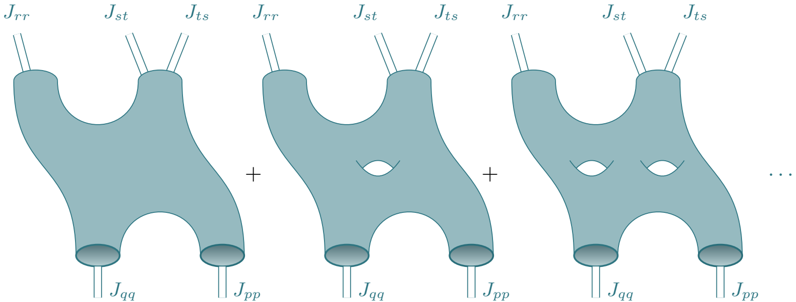
$$\lambda \prod_{\ell \in \mathbb{Z}_4} \delta_{a_{\ell+1}b_\ell}$$

EXPANSION IN BOUNDARY GRAPHS (ANY POTENTIAL)

$$
\begin{aligned}
W[J] = {} & \sum_{p} G_{|p|} J_{pp} + \frac{1}{2} \sum_{p,q} \Big(G_{|pq|} J_{pq} J_{qp} + G_{|p|q|} J_{pp} J_{qq} \Big) \\
& + \sum_{p,q,r} \Big(\frac{1}{3} G_{|pqr|} J_{pq} J_{qr} J_{rp} + \frac{1}{2} G_{|pq|r|} J_{pq} J_{qp} J_{rr} + \frac{1}{3!} G_{|p|q|r|} J_{pp} J_{qq} J_{rr} \Big) \\
& + \sum_{p,q,r,s} \Big(\frac{1}{4} G_{|pqrs|} J_{pq} J_{qr} J_{rs} J_{sp} + \frac{1}{3} G_{|pqr|s|} J_{pq} J_{qr} J_{rp} J_{ss} \\
& + \frac{1}{8} G_{|pq|rs|} J_{pq} J_{qp} J_{rs} J_{sr} + \frac{1}{4} G_{|p|q|rs|} J_{pp} J_{qq} J_{rs} J_{sr} + \frac{1}{4!} G_{|p|q|r|s|} J_{pp} J_{qq} J_{rr} J_{ss} \Big) \\
& + \sum_{p,q,r,s,t} \Big(\frac{1}{5} G_{|pqrst|} J_{pq} J_{qr} J_{rs} J_{st} J_{tp} + \frac{1}{4} G_{|p|qrst|} J_{pp} J_{qr} J_{rs} J_{st} J_{tq} \\
& + \frac{1}{2 \cdot 3} G_{|pq|rst|} J_{pq} J_{qp} J_{rs} J_{st} J_{tr} + \frac{1}{2^2 2!} G_{|p|qr|st|} J_{pp} J_{qr} J_{rq} J_{st} J_{ts} \\
& + \frac{1}{3!2} G_{|p|q|r|st|} J_{pp} J_{qq} J_{rr} J_{st} J_{ts} + \frac{1}{5!} G_{|p|q|r|s|t|} J_{pp} J_{qq} J_{rr} J_{ss} J_{tt} \Big) + \mathcal{O}(J^6)
\end{aligned}
$$

EXPANSION IN GENUS

$$G_{|p|q|r|st|}J_{pp}J_{qq}J_{rr}J_{st}J_{ts} = \sum_g G^{(g)}_{|p|q|r|st|}J_{pp}J_{qq}J_{rr}J_{st}J_{ts}$$



Coloured Tensor Models

- a quantum field theory for tensors $\varphi_{a_1 \dots a_D}$ and $\overline{\varphi}_{a_1 \dots a_D}$
- the indices transform under different representations of

$$H = U(N_1) \times U(N_2) \times \dots \times U(N_D)$$

- for $h \in H$, $U^{(a)} \in U(N_a)$,

$$\varphi_{a_1 a_2 \dots a_D} \xrightarrow{h} (\varphi')_{a_1 a_2 \dots a_D} = U^{(1)}_{a_1 b_1} U^{(2)}_{a_2 b_2} \dots U^{(D)}_{a_D b_D} \varphi_{b_1 b_2 \dots b_D}$$

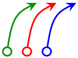
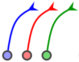
- the complex conjugate tensor $\overline{\varphi}_{a_1 a_2 \dots a_D}$ transforms as

$$\overline{\varphi}_{a_1 a_2 \dots a_D} \xrightarrow{h} (\overline{\varphi}')_{a_1 a_2 \dots a_D} = \overline{U}^{(1)}_{a_1 b_1} \overline{U}^{(2)}_{a_2 b_2} \dots \overline{U}^{(D)}_{a_D b_D} \overline{\varphi}_{b_1 b_2 \dots b_D}$$

- observables are invariants under $U(N_1) \times \dots \times U(N_D)$
- these invariants serve as interaction vertices

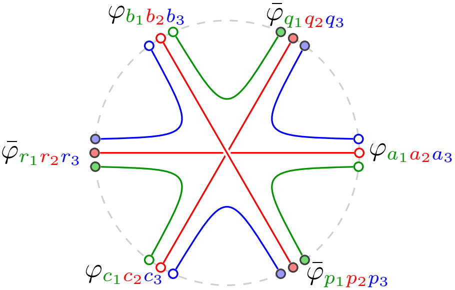
$$S[\varphi, \overline{\varphi}] = \sum_i \tau_i \mathrm{Tr}_{\mathcal{B}_i}(\varphi, \overline{\varphi}) = \mathrm{Tr}_{\mathcal{B}_2}(\overline{\varphi}, \varphi) + \sum_\alpha \lambda_\alpha \mathrm{Tr}_{\mathcal{B}_\alpha}(\overline{\varphi}, \varphi)$$

- traces $\mathrm{Tr}_{\mathcal{B}}$ are indexed by bipartite D-coloured graphs $\mathcal{B}$
- In $D = 3$ colours we associate

$\varphi_{a_1a_2a_3} \longmapsto$  **and** $\bar{\varphi}_{p_1p_2p_3} \longmapsto$  (reversed order!)

- Contract: $\xrightarrow{1} = \delta_{a_1p_1}$ $\xrightarrow{2} = \delta_{a_2p_2}$ $\xrightarrow{3} = \delta_{a_3p_3}$

Example:



$$\mathrm{Tr}_{K_3}(\varphi, \bar{\varphi}) = \sum_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q},\mathbf{r}} \left(\bar{\varphi}_{r_1r_2r_3}\bar{\varphi}_{q_1q_2q_3}\bar{\varphi}_{p_1p_2p_3}\right) \cdot \left(\delta_{a_1p_1}\delta_{a_2r_2}\delta_{a_3q_3}\delta_{b_1q_1}\delta_{b_2p_2}\delta_{b_3r_3} \delta_{c_1r_1}\delta_{c_2q_2}\delta_{c_3p_3}\right) \cdot \left(\varphi_{a_1a_2a_3}\varphi_{b_1b_2b_3}\varphi_{c_1c_2c_3}\right)$$

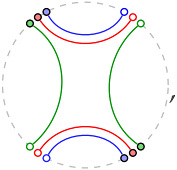
Feynman diagrams

Choose an action, for instance,

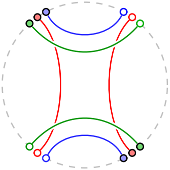
$$S[\varphi,\bar{\varphi}] = \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi}) + \lambda\big(\mathrm{Tr}_{\mathcal{V}_1}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_2}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_3}(\varphi,\overline{\varphi})\big)$$

and

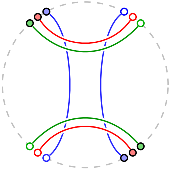
$\mathcal{V}_1 =$



, $\mathcal{V}_2 =$



, $\mathcal{V}_3 =$



,

$$Z[J,\bar{J}] = \frac{\int \mathcal{D}\varphi\mathcal{D}\overline{\varphi}\, \mathrm{e}^{\mathrm{Tr}_{\mathcal{B}_2}(\bar{J}\varphi)+\mathrm{Tr}_{\mathcal{B}_2}(\bar{\varphi}J)-N^2S[\varphi,\overline{\varphi}]}}{\int \mathcal{D}\varphi\mathcal{D}\overline{\varphi}\, \mathrm{e}^{-N^2S[\varphi,\overline{\varphi}]}},$$

with $\mathrm{Tr}_{\mathcal{B}_2} \leftrightarrow$

- Write  for Wick's contractions w.r.t. the Gaußian measure

$$\mathrm{d}\mu_C(\varphi,\overline{\varphi}) := \mathcal{D}\varphi\mathcal{D}\overline{\varphi}\mathrm{e}^{-N^2S_0[\varphi,\overline{\varphi}]} := \prod_a \frac{\mathrm{d}\varphi_a\mathrm{d}\overline{\varphi}_a}{2\pi\mathrm{i}}\mathrm{e}^{-N^2\mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi})}$$

$$\int \mathrm{d}\mu_C(\varphi,\overline{\varphi})\varphi_{\mathbf{a}}\overline{\varphi}_{\mathbf{p}} = C(\mathbf{a},\mathbf{p}) = \delta_{\mathbf{ap}} = \mathbf{a}$$

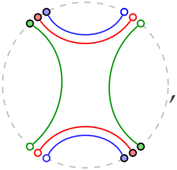
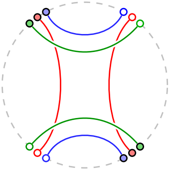
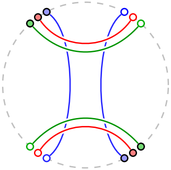
$\mathbf{p}$

Feynman diagrams

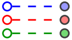
Choose an action, for instance,

$$S[\varphi,\bar{\varphi}] = \mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi}) + \lambda(\mathrm{Tr}_{\mathcal{V}_1}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_2}(\varphi,\overline{\varphi}) + \mathrm{Tr}_{\mathcal{V}_3}(\varphi,\overline{\varphi}))$$

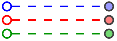
and

$\mathcal{V}_1 =$  , $\mathcal{V}_2 =$  , $\mathcal{V}_3 =$  ,

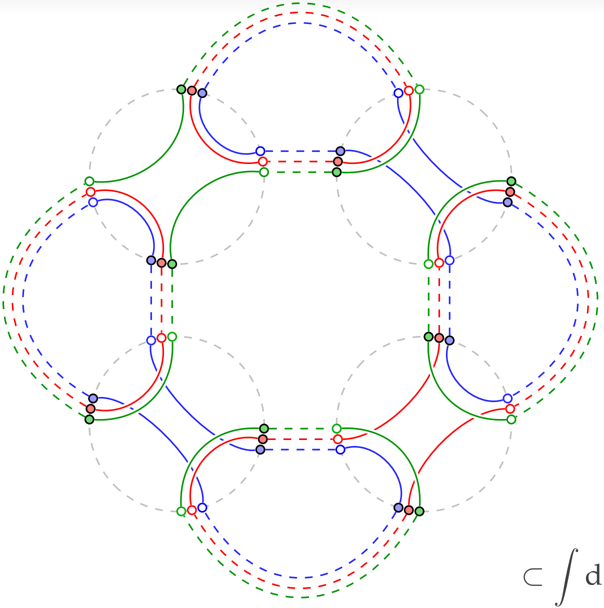
$$Z[J,\bar{J}] = \frac{\int \mathcal{D}\varphi\mathcal{D}\overline{\varphi}\, \mathrm{e}^{\mathrm{Tr}_{\mathcal{B}_2}(\bar{J}\varphi)+\mathrm{Tr}_{\mathcal{B}_2}(\bar{\varphi}J)-N^2S[\varphi,\overline{\varphi}]}}{\int \mathcal{D}\varphi\mathcal{D}\overline{\varphi}\, \mathrm{e}^{-N^2S[\varphi,\overline{\varphi}]}},$$ with $\mathrm{Tr}_{\mathcal{B}_2} \leftrightarrow$ 

- Write  for Wick's contractions w.r.t. the Gaußian measure

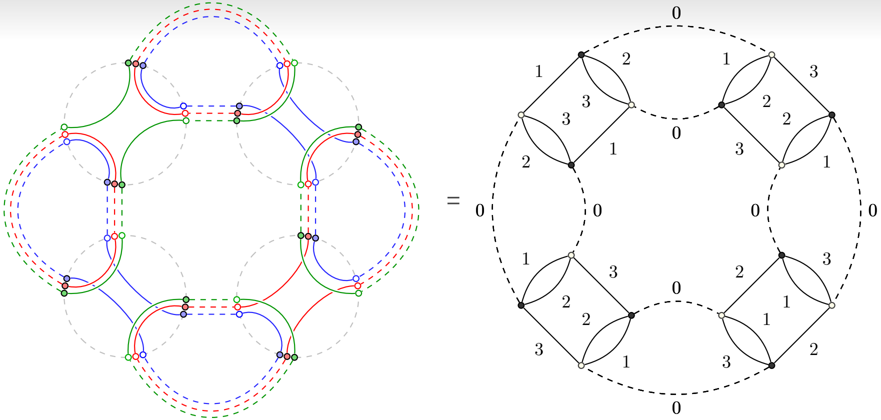
$$\mathrm{d}\mu_C(\varphi,\overline{\varphi}) := \mathcal{D}\varphi\mathcal{D}\overline{\varphi}\mathrm{e}^{-N^2S_0[\varphi,\overline{\varphi}]} := \prod_a \frac{\mathrm{d}\varphi_a\mathrm{d}\overline{\varphi}_a}{2\pi\mathrm{i}}\mathrm{e}^{-N^2\mathrm{Tr}_{\mathcal{B}_2}(\varphi,\overline{\varphi})}$$

$$\int \mathrm{d}\mu_C(\varphi,\overline{\varphi})\varphi_{\mathbf{a}}\overline{\varphi}_{\mathbf{p}} = C(\mathbf{a},\mathbf{p}) = \delta_{\mathbf{ap}} = \mathbf{a}$$  $\mathbf{p}$

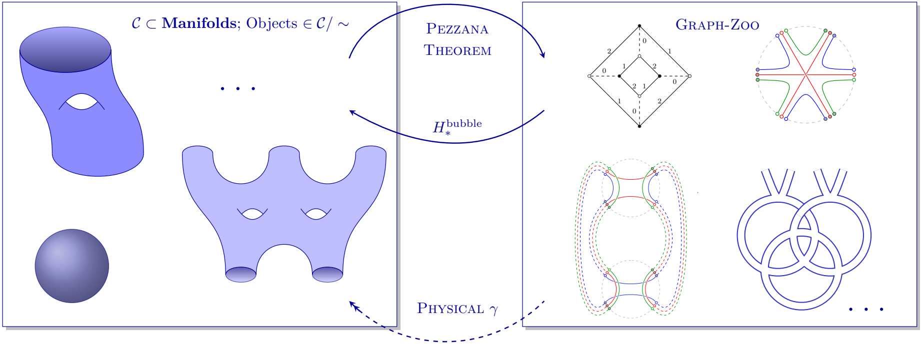
- Example. An $\mathcal{O}(\lambda^4)$-contribution (vacuum sector)



$$\subset \int \mathrm{d}\mu_C(\mathcal{V}_1\mathcal{V}_2[\mathcal{V}_3]^2)$$



- Feynman graphs $\mathcal{G}$ are $(D+1)$-coloured: amplitude controlled by geometric data [Gurău, 11]: supports large-N expansion
- According to crystallization theory [Pezzana, '74]— alternatively GEMs — these graphs represent PL D-manifolds



$\mathcal{C} \subset$ **Manifolds**; Objects $\in \mathcal{C}/\sim$

PEZZANA THEOREM

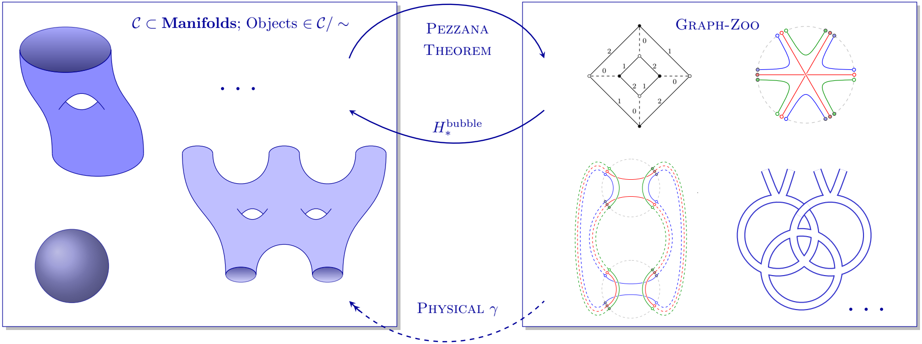
H_*^{bubble}

PHYSICAL γ

GRAPH-ZOO

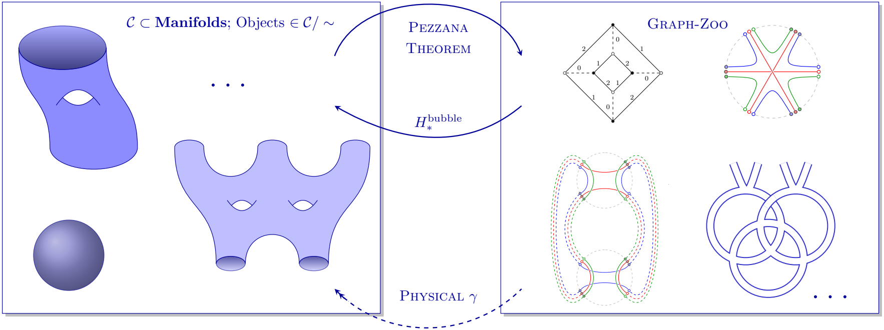
[C. I. Pérez-S. arXiv:1608.00246]

$$\begin{tikzcd}
& \mathfrak{F}^0_D(V(\Phi)) \arrow[dl] \arrow[dr, dashed, "\gamma"] & \\
\mathsf{Grph}_{c,D+1} \arrow[rr, "\Delta"] & & \mathcal{C}/\sim
\end{tikzcd}$$



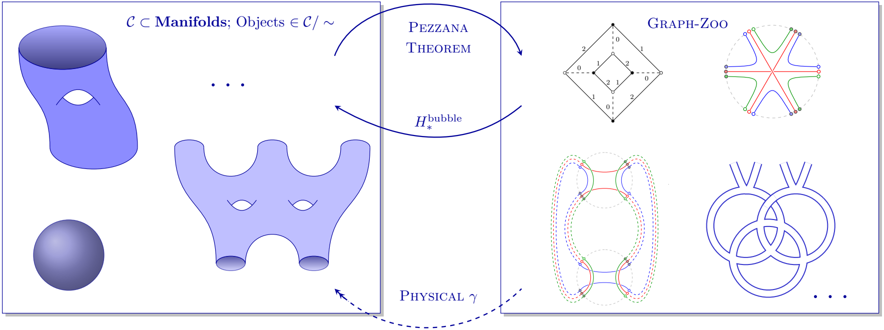
[C. I. Pérez-S. arXiv:1608.00246]

$$\begin{tikzcd}
& \mathfrak{F}^0_D(V(\Phi)) \arrow[dl, hook'] \arrow[dr, dashed, "\gamma"] & \\
\mathsf{Grph}_{c,D+1} \arrow[rr, "\Delta"] & & \mathcal{C}/\sim
\end{tikzcd}$$



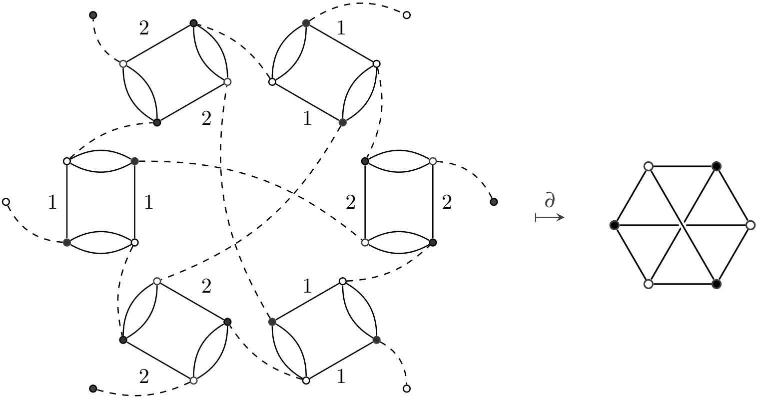
[C. I. Pérez-S. arXiv:1608.00246]

$$\begin{tikzcd}
 & \mathfrak{F}_2^0((\varphi\bar{\varphi})^2) \arrow[dl, hook] \arrow[dr, dashed, "\gamma"] & \\
\mathsf{Grph}_{c,2+1} \arrow[rr, "\Delta"] & & \mathsf{Riem}_{\mathrm{cco}}
\end{tikzcd}$$



[C. I. Pérez-S. arXiv:1608.00246]

$$\begin{tikzcd}
& \mathfrak{F}_2((\varphi\bar{\varphi})^2) \arrow[dl, hook] \arrow[dr, dashed, "\xi"] & \\
\dot{\cup}_{k=0}^{\infty} \mathsf{Grph}^{(2k)}_{\text{c},2+1} \arrow[rr, "\Delta"] & & \text{2-Cob}
\end{tikzcd}$$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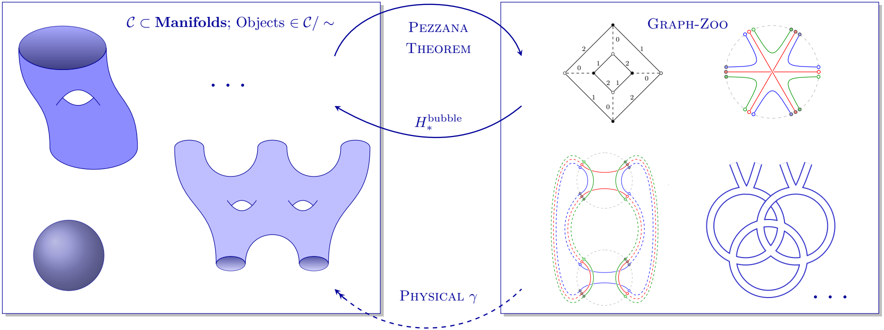


$\mathcal{C} \subset$ **Manifolds**; Objects $\in \mathcal{C}/\sim$

PEZZANA THEOREM

GRAPH-ZOO

H_*^{bubble}

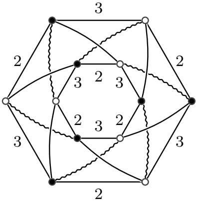
PHYSICAL γ

- $|\Delta\partial\mathcal{G}| \cong \partial|\Delta\mathcal{G}|$

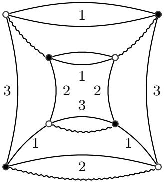
$$\begin{tikzcd}
 & \mathfrak{F}_D(\varphi^4_{\mathsf{m}}) \arrow[dl, "\partial"'] \arrow[dr, dashed] & \\
\mathsf{Grph}^{(0)}_{\mathsf{c},D} \arrow[rr, "\Delta"] & & (D-1)\text{-manifolds}_{\mathrm{PL}}
\end{tikzcd}$$

Once one has found a crystallization of the boundary, e.g. here in $D = 4$,

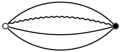
$\mathcal{P} =$



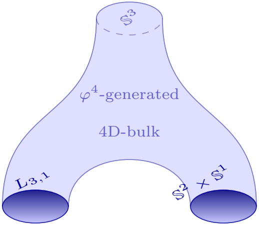
$\mathcal{C} =$



$\mathcal{M} =$



Then $G^{(\mathcal{N})}_{\mathcal{C}|\mathcal{P}|\mathcal{M}}$ is interpreted as



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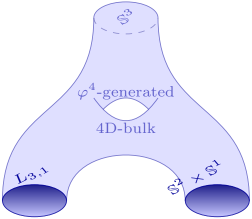

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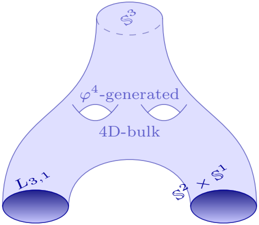
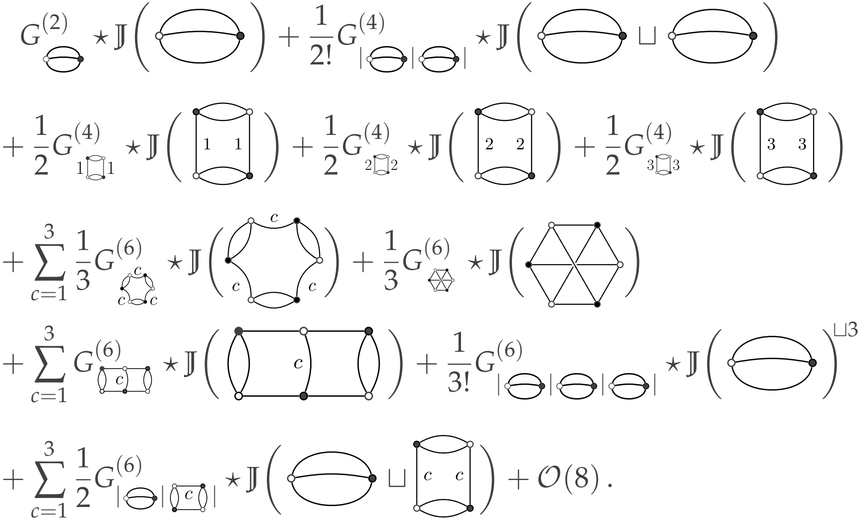


$+ \ldots$

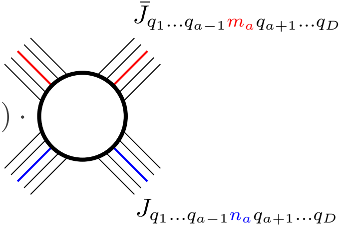
EXPANSION OF $W[J,\bar{J}]$ IN $D = 3$



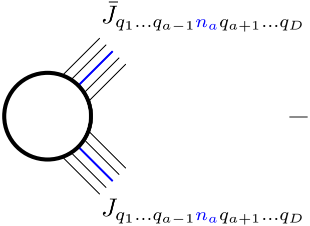
Ward Takahashi Identity

- For matrix models [Disertori-Gurău-Magnen-Rivasseau]: path integral measure is $\mathrm{U}(N)$-invariant. This implies relations between $G^{(k)}$ and $G^{(k+2)}$. For tensors models ($\supset$ matrix models) [D. Ousmane]:

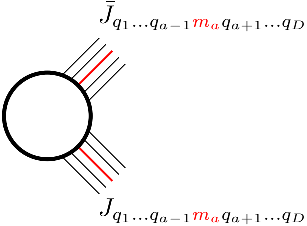
$$\sum_{p_i \in \mathbb{Z}} \left(E_{p_1 \ldots m_a \ldots p_D} - E_{p_1 \ldots n_a \ldots p_D}\right) \cdot$$



$$=$$



$$-$$



WARD TAKAHASHI IDENTITY

- For matrix models [Disertori-Gurău-Magnen-Rivasseau]: path integral measure is $\mathrm{U}(N)$-invariant. This implies relations between $G^{(k)}$ and $G^{(k+2)}$. For tensors models ($\supset$ matrix models) [D. Ousmane]:
- Terms annihilated by the difference $\Delta E_{m_a n_a}$ can be found using the expansion of $\log Z[J,\bar{J}]$ in boundary graphs $\Rightarrow$ A new full Ward-Takahashi Identity [C. I. Pérez-S. arXiv:160$^{8}_{9}$. ¿ ?]
- Leads to integro-differential equations

Outlook:

- Add matter: Gauge fields, for matrix models. E.g. NCG-setting via reps. of graphs on the category of finite spectral triples
- Combine Ward Identity with Schwinger-Dyson equations.

REFERENCES

- H. Grosse and R. Wulkenhaar
  Commun. Math. Phys. **329** (2014) 1069
  [arXiv:1205.0465 [math-ph]].
- R. Gurău
  Commun. Math. Phys. **304**, 69 (2011)
  [arXiv:0907.2582 [hep-th]].
- D. Ousmane Samary, C. I. Pérez-Sánchez, F. Vignes-Tourneret and R. Wulkenhaar Class. Quant. Grav. **32** (2015) 17, 175012
  [arXiv:1411.7213 [hep-th]].
- Carlos I. Pérez-Sánchez
  arXiv:1608.00246 [math-ph]

Thank you for your attention