Fuzzy spaces and applications

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outline

1. Lecture I: basics
   - outline, motivation
   - Poisson structures, symplectic structures and quantization
   - basic examples of fuzzy spaces
     ($S^2_N,\ T^2_N,\ \mathbb{R}^4_\theta$ etc.)
   - quantized coadjoint orbits ($\mathbb{C}P^n_N$)
   - generic fuzzy spaces; fuzzy S^4_N, squashed $\mathbb{C}P^2$ etc.
   - counterexample: Connes torus
2. Lecture II: developments
   - coherent states on fuzzy spaces (Perelomov)
   - symbols and operators, semi-class limit, visualization
   - uncertainty, UV/IR regimes on S^2_N etc.
3. Lecture III: applications
   - NCFT on fuzzy spaces: scalar fields & loops
   - NC gauge theory from matrix models
   - IKKT model

- emergent gravity on S^4_N

literature:
These lectures will loosely follow the following:

- introductory review:

  H.S., “Noncommutative geometry and matrix models”. arXiv:1109.5521
- H. C. Steinacker, “String states, loops and effective actions in noncommutative field theory and matrix models,” [arXiv:1606.00646 [hep-th]].
- L. Schneiderbauer and H. C. Steinacker, “Measuring finite Quantum Geometries via Quasi-Coherent States,” [arXiv:1601.08007 [hep-th]].

Further related useful literature is e.g.

- J. Madore, “The Fuzzy sphere,” Class. Quant. Grav. **9**, 69 (1992).
- Richard J. Szabo, “Quantum Field Theory on Noncommutative Spaces”

  arXiv:hep-th/0109162v4
- M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” [hep-th/0106048].
- H. Steinacker, “Emergent Geometry and Gravity from Matrix Models: an Introduction,” [arXiv:1003.4134 [hep-th]].
- N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A large-N reduced model as superstring,” [arXiv:hep-th/9612115].

1 Lecture I: basics

Motivation, scope

gravity $\leftrightarrow$ quantum mechanics

general relativity (1915) established at low energies, long distances

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$$

space-time: pseudo-Riemannian manifold $(\mathcal{M}, g)$, dynamical metric $g_{\mu\nu}$ describes gravity through the Einstein equations.

is incomplete (singularities)

no natural quantization (non-renormalizable)

Q.M. & G.R. $\Rightarrow$ break-down of classical space-time below $L_{Pl} = \sqrt{\hbar G/c^3} = 10^{-33}cm$

classical concept of space-time as manifold physically not meaningful at scales $(\Delta x)^2 \leq L_{Pl}^2$

$\rightarrow$ expect quantum structure of space-time at Planck scale

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standard argument: Consider an object of size Δx.

Heisenbergs uncertainty relation $\Rightarrow$momentum is uncertain by $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$,

i.e. momentum takes values up to at least $\Delta p = \frac{\hbar}{2\Delta x}$.

$\Rightarrow$ it has an energy or mass $mc^2 = E \geq \Delta pc = \frac{\hbar c}{2\Delta x}$

G.R. $\Rightarrow$ $\Delta x \geq R_{\text{Schwarzschild}} \sim 2G\frac{E}{c^4} \geq \frac{\hbar G}{c^3 \Delta x}$

$\Rightarrow$ $(\Delta x)^2 \geq \hbar G/c^3 = L_{Pl}^2$

more precise version:

(Doplicher Fredenhagen Roberts 1995 hep-th/0303037)

1.1 NC geometry

replace commutative algebra of functions $\rightarrow$ NC algebra of "functions"

(cf. Gelfand-Naimark theorem)

inspired by quantum mechanics: quantized phase space

$$[X^\mu, P_\nu] = i\hbar\delta^\mu_\nu$$

$\rightarrow$ area quantization $\Delta X^\mu \Delta P_\mu = \frac{\hbar}{2}$ (Bohr-Sommerfeld quantization!)

NCG: not just NC algebra, but extra structure which defines the geometry

many posssibilities

- Connes: (math) spectral triples
- here: alternative approach, motivated by physics, string theory, matrix models

1.2 Fuzzy spaces

Definition 1.1. *Fuzzy space = noncommutative space* $\mathcal{M}_N \hookrightarrow \mathbb{R}^D$ *with intrinsic UV cutoff, finitely many d.o.f. per unit volume*

similar mathematics & concepts as in Q.M., but applied to **configuration space** (space-time) instead of phase space

$$[X^\mu, X^\nu] = i|\theta^{\mu\nu}|$$

$\rightarrow$ typically **quantized symplectic space**
$\rightarrow$ area quantization $\Delta X^\mu \Delta X^\nu \geq \frac{\theta^{\mu\nu}}{2}$, finitely many d.o.f per unit volume
note:

- geometry from embedding in target space $\mathbb{R}^n$

  distinct from spectral triple approach (Connes)
- arises in string theory from D0 branes in background flux ("dielectric branes")
- arises as nontrivial vacuum solutions in Yang-Mills gauge theory with large rank ("fuzzy extra dimensions")
- condensed matter physics in strong magnetic fields (quantum Hall effect, monopoles (?) ...)

goal:

- formulate physical models (QFT) on fuzzy spaces

  study UV divergences in QFT (UV/IR mixing)
- find dynamical quantum theory of fuzzy spaces ($\rightarrow$ quantum gravity ?!)

1.3 Poisson / symplectic spaces & quantization

$\{.,.\}: \quad \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ ... Poisson structure if

$$\begin{aligned} \{f,g\} + \{g,f\} &= 0, \qquad \text{anti-symmetric} \\ \{f\cdot g, h\} &= f\cdot\{g,h\} + \{f,h\}\cdot g \qquad \text{Leibnitz rule / derivation,} \\ \{f,\{g,h\}\} + \text{cyclic} &= 0 \qquad \text{Jacobi identity} \end{aligned}$$

$\leftrightarrow$ tensor field $\theta^{\mu\nu}(x)\partial_\mu \wedge \partial_\nu$ with

$$\theta^{\mu\nu} = -\theta^{\nu\mu}, \quad \theta^{\mu\mu'}\partial_{\mu'}\theta^{\nu\rho} + \text{cyclic} = 0$$

assume $\theta^{\mu\nu}$ non-degenerate
Then $\boxed{exercise\ 1}$:

$$\begin{aligned} \omega &:= \tfrac{1}{2}\theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu \qquad \in \Omega^2\mathcal{M} \quad \text{closed,} \\ d\omega &= 0 \end{aligned}$$

... symplectic form (=a closed non-degenerate 2-form)

<u>examples:</u>

- cotanget bundle: let $\mathcal{M}$ ... manifold, local coords x^i

  $T^*\mathcal{M}$ ... bundle of 1-forms $p_i(x)dx^i$ over $\mathcal{M}$

  local coords on $T^*\mathcal{M}$: x^i, p_j

  at point $(x^i, p_j) \in T^*\mathcal{M}$, choose the one-form $\theta = p_i dx^i$. This defines a canonical (tautological) 1-form θ on $T^*\mathcal{M}$.

  The symplectic form is defined as $\omega = d\theta = dp_i dx^i$

- any orientable 2-dim. manifold

  ω ... any 2-form, e.g. volume-form

  e.g. 2-sphere S^2: let $\omega =$ unique $SO(3)$ -invariant 2- form

<u>Darboux theorem:</u>
suppose that ω is a symplectic 2-form on a $2n$- dimensional manifold $\mathcal{M}$. for every $p \in \mathcal{M}$ there is a local neighborhood with coordinates $x^\mu, y^\mu, \ \mu = 1, ..., n$ such that

$$\omega = dx^1 \wedge dy^1 + ... + dx^n \wedge dy^n = d\theta.$$

so all symplectic manifolds with equal dimension are locally isomorphic

1.4 Quantized Poisson (symplectic) spaces

$(\mathcal{M}, \theta^{\mu\nu}(x))$ ... $2n$-dimensional manifold with Poisson structure

Its quantization $\mathcal{M}_\theta$ is given by a NC (operator) algebra $\mathcal{A}$ and a (linear) quantization map $\mathcal{Q}$

$$\begin{aligned} \mathcal{Q}: \ \mathcal{C}(\mathcal{M}) &\rightarrow \mathcal{A} \subset End(\mathcal{H}) \\ f(x) &\mapsto \hat{f} \end{aligned}$$

such that

$$\begin{aligned} (\hat{f})^\dagger &= \widehat{f^*} \\ \hat{f}\,\hat{g} &= \widehat{fg} + o(\theta) \\ [\hat{f}, \hat{g}] &= i\widehat{\{f,g\}} + o(\theta^2) \end{aligned}$$

or equivalently

$$\frac{1}{\theta}\big([\hat{f}, \hat{g}] - i\widehat{\{f,g\}}\big) \rightarrow 0 \ \text{ as } \ \theta \rightarrow 0.$$

here $\mathcal{H}$ ... separable Hilbert space

$\mathcal{Q}$ should be an isomorphism of vector spaces (at least at low scales), such that (“nice“) $\Phi \in End(\mathcal{H}) \quad \leftrightarrow \quad$ quantized function on $\mathcal{M}$

cf. correspondence principle

we will assume that the Poisson structure is non-degenerate, corresponding to a symplectic structure ω.

Then the trace is related to the integral as follows:

$$\begin{aligned} (2\pi)^n \mathrm{Tr}\, \mathcal{Q}(\phi) &\sim \int \tfrac{\omega^n}{n!}\phi = \int d^{2n}x\, \rho(x)\, \phi(x) \\ \rho(x) &= \mathrm{Pfaff}\, (\theta^{-1}_{\mu\nu}) = \sqrt{\det \theta^{-1}_{\mu\nu}} \ \ldots \ \text{symplectic volume} \end{aligned}$$

(recall that $\frac{\omega^n}{n!}$ is the Liouville volume form. This will be justified below)

Interpretation:

$$\rho(y) = \sqrt{\det \theta^{-1}_{\mu\nu}} =: \Lambda^{2n}_{NC}$$

where Λ_{NC} can be interpreted as “local” scale of noncommutativity.

in particular: $\dim(\mathcal{H}) \sim \mathrm{Vol}(\mathcal{M})$, (cf. Bohr-Sommerfeld)

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examples & remarks:

- **Quantum Mechanics**:

  phase space $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 = T^*\mathbb{R}^3$, coords (p_i, q^i),

  Poisson bracket $\{q^i, p_j\} = \delta^j_i$ replaced by canonical commutation relations $[Q^i, P_j] = i\hbar\delta^i_j$

- reformulate same structure as $\mathbb{R}^2_\hbar$ = **Moyal-Weyl quantum plane**

  $X^\mu = \begin{pmatrix} Q \\ P \end{pmatrix}$, Heisenberg C.R.

  $$[X^\mu, X^\nu] = i\theta^{\mu\nu}\,\mathbf{1}, \qquad \mu,\nu = 1,...,2, \qquad \theta^{\mu\nu} = \hbar\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  $\mathcal{A} \subset End(\mathcal{H})$ ... functions on $\mathbb{R}^2_\hbar$

  uncertainty relations $\Delta X^\mu \Delta X^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|$

  Weyl-quantization: Poisson structure $\{x^\mu, x^\nu\} = \theta^{\mu\nu}$

  $$\mathcal{Q}: \quad L^2(\mathbb{R}^2) \to \mathcal{A} \subset \mathcal{L}(\mathcal{H}), \quad \text{(Hilbert-Schmidt operators)}$$

  $$\phi(x) = \int d^2k\, e^{ik_\mu x^\mu}\hat{\phi}(k) \quad \mapsto \quad \int d^2k\, e^{ik_\mu X^\mu}\hat{\phi}(k) =: \Phi(X) \in \mathcal{A}$$

  respects translation group.

  interpretation:

  $X^\mu \quad \in \mathcal{A} \cong End(\mathcal{H})$ ... quantiz. coord. function on $\mathbb{R}^2_\hbar$

  $\Phi(X^\mu) \in End(\mathcal{H})$ ... observables (functions) on $\mathbb{R}^2_\hbar$

- $\mathcal{Q}$ not unique, not Lie-algebra homomorphism

  (Groenewold-van Hove theorem)

- existence, precise def. of quantization non-trivial, $\exists$ various versions:

  - formal (as formal power series in θ):
    always possible (Kontsevich 1997) but typically not convergent
  - strict (= as C^* algebra resp. in terms of operators on $\mathcal{H}$),
  - etc.

  need strict quantization (operators)

  $\exists$ existence theorems for Kähler-manifolds (Schlichenmaier etal),

  almost-Kähler manifolds (= very general) (Uribe etal)

- semi-classical limit:

  work with commutative functions (de-quantization map),

  replace commutators by Poisson brackets

i.e. replace

$$\begin{array}{rcl} \hat{F} & \to & f = \mathcal{Q}^{-1}(F) \\ [\hat{F}, \hat{G}] & \to & i\{f, g\} \qquad (+O(\theta^2), \quad \text{drop}) \end{array}$$

i.e. keep only leading order in θ

1.5 Embedded non-commutative (fuzzy) spaces

Consider a symplectic manifold embedded in target space,

$$x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D, \qquad a = 1, \ldots, D$$

(not necessarily injective)
and some quantization $\mathcal{Q}$ as above. Then define

$$X^a := \mathcal{Q}(x^a) = X^{a\dagger} \in End(\mathcal{H}) \ .$$

If $\mathcal{M}$ is compact, these will be finite-dimensional matrices, which describe **quantized embedded symplectic space = fuzzy space**.

Definition 1.2. *A fuzzy space is defined in terms of a set of D hermitian matrices $X^a \in End(\mathcal{H})$, $a = 1, \ldots, D$, which admits an approximate "semi-classical" description as quantized embedded symplectic space with $X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$.*

aim: develop a systematic procedure to extract the effective geometry, formulate & study physical models on these.

1.6 The fuzzy sphere

1.6.1 classical S^2

$$\begin{array}{rcl} x^a : \ S^2 & \hookrightarrow & \mathbb{R}^3 \\ x^a x^a & = & 1 \end{array}$$

algebra $\mathcal{A} = \mathcal{C}^\infty(S^2)$ ... spanned by spherical harmonics Y^l_m = polynomials of degree l in x^a
choose $SO(3)$-invariant symplectic form ω, normalized as $\int \omega = 2\pi N$

1.6.2 fuzzy S^2_N

(Hoppe 1982, Madore 1992)

S^2 compact $\Rightarrow$ $\mathcal{H} = \mathbb{C}^N$, $\mathcal{A}_N = End(\mathcal{H}) = Mat(N, \mathbb{C})$
would like to preserve rotational symmetry $SO(3)$

$\mathfrak{su}(2)$ action on $\mathcal{A}_N$:

Let J^a ... generators of $\mathfrak{su}(2)$,

$$[J^a, J^b] = i\epsilon^{abc} J^c$$

Let $\pi_{(N)}(J^a)$ ... $N-$ dim irrep of $\mathfrak{su}(2)$ on $\mathcal{H} = \mathbb{C}^N$ (spin $j = \frac{N-1}{2}$)
Define

$$\begin{array}{rcl} \mathfrak{su}(2) \times \mathcal{A}_N & \rightarrow & \mathcal{A}_N \\ (J^a, \phi) & \mapsto & [\pi_{(N)}(J^a), \phi] \end{array}$$

decompose $\mathcal{A}_N$ into irreps of $SO(3)$:

$$\begin{array}{rcl} \mathcal{A}_N = \mathrm{Mat(N, \mathbb{C})} \cong (\mathrm{N}) \otimes (\bar{\mathrm{N}}) & = & (1) \ \oplus (3) \ \oplus ... \oplus (2N-1) \\ & =: & \{\hat{Y}^0_0\} \oplus \{\hat{Y}^1_m\} \oplus ... \oplus \{\hat{Y}^{N-1}_m\}. \end{array}$$

... fuzzy spherical harmonics; UV cutoff in angular momentum!
Introduce Hilbert space structure on $\mathcal{A}_N = \mathrm{Mat(N, \mathbb{C})}$ by

$$(F, G) := \frac{4\pi}{N} \mathrm{Tr}(F^\dagger G)$$

corresponds to $L^2(S^2)$ with $(f, g) := \int_{S^2} f^* g$
normalize the $\hat{Y}^l_m$ such that ONB,

$$(\hat{Y}^l_m, \hat{Y}^{l'}_{m'}) = 4\pi \delta^{ll'} \delta_{mm'}$$

quantization map:

$$\begin{array}{rrcl} \mathcal{Q}: & \mathcal{C}(S^2) & \rightarrow & \mathcal{A}_N \\ & Y^l_m & \mapsto & \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases} \end{array}$$

satisfies $\mathcal{Q}(f^*) = \mathcal{Q}(f)^\dagger$

embedding functions want $X^a \sim x^a$

note: $x^i: \ S^2 \hookrightarrow \mathbb{R}^3$ are spin 1 harmonics, $Y^1_{\pm 1} = x^1 \pm ix^2$ and $Y^1_0 = x^3$.
Hence quantization given by $\hat{Y}^1_{\pm 1} = X^1 \pm iX^2$ and $\hat{Y}^1_0 = X^3$, i.e.

$$X^a := \mathcal{Q}(x^a) = C_N \, \pi_{(N)}(J^a)$$

for some constant C_N (unique spin 1 irrep).
It follows

$$[X^a, X^b] = i\, C_N\, \varepsilon_{abc} X^c$$

fix radius to be 1,

$$\sum_{a=1}^{3} (X^a)^2 = C_N^2 J^a J^a = C_N \frac{N^2-1}{4} \mathbf{1},$$

cf. quadratic Casimir, implies

$$C_N = 2/\sqrt{N^2-1} \approx \frac{2}{N}.$$

correspondence principle $\to$ Poisson structure

$$\{x^a, x^b\} = C_N\, \varepsilon_{abc}\, x^c \approx \frac{2}{N} \varepsilon_{abc}\, x^c$$

which is of order $\theta \sim 2/N$.
corresponds to $SU(2)$-invariant symplectic form

$$\omega = \frac{N}{4} \varepsilon_{abc} x^a dx^b dx^c \quad =: N\omega_1$$

on S^2 with $\int \omega = 2\pi N$.
(unique closed and $SO(3)$ invariant volume form)

$\boxed{Exercise\ 2}$: check this by introducing local coordinates x^1, x^2 near north pole.
at north pole (NP): $\{x^1, x^2\} = \frac{2}{N}$
$\Rightarrow$ symplectic structure $\theta_{12}^{-1} = \frac{N}{2}$ at NP

<u>therefore:</u>

$$\boxed{S_N^2 \text{ is quantization of } (S^2, N\omega_1)}$$

<u>integral:</u> $(2\pi)\mathrm{Tr}(\mathcal{Q}(f)) = \int_{S^2} \omega f$
(only $\hat{Y}_0^0 \sim \mathbf{1}$ contributes).

$\exists$ inductive sequences of fuzzy spheres

$$\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1} \hookrightarrow \ldots \hookrightarrow \mathcal{A} = \mathcal{C}^\infty(S^2)$$

respecting norm and group structure (not algebra).

Realize $\hat{Y}^l_m = P^l_m(X)$ as totally symmetrized polynomials. Clearly the generators X^a commute up to $\frac{1}{N}$ corrections, hence $\mathcal{Q}(fg) \to \mathcal{Q}(f)\mathcal{Q}(g)$ for $N \to \infty$, for fixed quantum numbers. Thus

$$\begin{aligned}\mathcal{Q}(fg) &= \mathcal{Q}(f)\mathcal{Q}(g) \ + O(\tfrac{1}{N}),\\ \mathcal{Q}(i\{f,g\}) &= [\mathcal{Q}(f),\mathcal{Q}(g)] \ + O(\tfrac{1}{N^2})\end{aligned}$$

for fixed angular momenta $\ll N$.
For a fixed S^2_N. the relation with the classical case is only justified for low angular momenta, consistent with a Wilsonian point of view. (One should then only ask for estimates for the deviation from the classical case.)

example: consider the coordinate "function"

$$X^3 = \frac{2}{\sqrt{N^2-1}}\mathrm{diag}((N-1)/2,(N-1)/2-1,...,-(N-1)/2)$$

normalization such that the spectrum is essentially dense from -1 to 1.

local description: near "north pole" $X^3 \approx 1, \qquad X^1 \approx X^1 \approx 0$

$$\begin{aligned}X^3 &= \sqrt{1-(X^1)^2-(X^2)^2}\\ [X^1,X^2] &= \tfrac{i}{\sqrt{C_N}}X^3 =: \theta^{12}(X) \ \approx \ \tfrac{2i}{N}\end{aligned}$$

cf. Heisenberg algebra!

quantum cell $\Delta A = \Delta X^1 \Delta X^2 \geq \frac{1}{N}$, total area $N\Delta A \sim 1$

S^2_N consists of N quantum cells

$\boxed{Exercise\ 3:}$ Work out the "Jordan-Schwinger" ("2nd quantized") realization for the fuzzy sphere, i.e. define

$$X^i := a^+_\alpha (\sigma^i)^\alpha_\beta a^\beta, \qquad \alpha = 1,2$$

for bosonic creation- and anihilation operators $[a^\alpha, a^+_\beta] = \delta^\alpha_b$ acting on the bosonic Fock space $\mathcal{F} = \oplus_N \mathcal{F}_N, \qquad \mathcal{F}_N = \underbrace{a^+ ... a^+}_{N times}|0\rangle$.
Show that the X^i can be restricted to the N-particle sector $\mathcal{F}_N$ specified by $X^i X_i \sim \hat{N} = a^+_\alpha a^\alpha = const$, and satisfy on $\mathcal{F}_N$ the relations of a fuzzy sphere S^2_N.

1.7 Metric structure of the fuzzy sphere

$SO(3)$ symmetry $\Rightarrow$ expect "round sphere"

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metric encoded in NC Laplace operator

$$\Box: \ \mathcal{A} \to \mathcal{A}, \qquad \Box\phi = [X^a,[X^b,\phi]]\delta_{ab}$$

$SO(3)$ invariant: $\Box(g \triangleright \phi) = g \triangleright (\Box\phi) \quad \Rightarrow \ \Box\hat{Y}^l_m = c_l \hat{Y}^l_m$

note: $\Box = C_N^2 \, J^a J^a$ on $\quad \mathcal{A} \cong (N) \otimes (\bar{N}) \cong (1) \oplus (3) \oplus ... \oplus (2N-1)$

$$\Rightarrow \qquad \boxed{\Box\hat{Y}^l_m = C_N^2 \, l(l+1)\hat{Y}^l_m}$$

spectrum identical with classical case $\Delta_g\phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi)$

up to cutoff

$$\boxed{\Rightarrow \ \text{effective metric of } \Box = \text{round metric on } S^2}$$

1.8 Fuzzy torus T^2_N

def. $U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & & \dots & 0 & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i \frac{1}{N}} & & & \\ & & e^{2\pi i \frac{2}{N}} & & \\ & & & \ddots & \\ & & & & e^{2\pi i \frac{N-1}{N}} \end{pmatrix}$ satisfy

$$\begin{aligned} UV &= qVU, \qquad U^N = V^N = 1, \qquad q = e^{2\pi i \frac{1}{N}} \\ [U,V] &= (q-1)VU \end{aligned}$$

generate $\mathcal{A} = \mathrm{Mat}(\mathrm{N},\mathbb{C})$ ... quantiz. algebra of functions on T^2_N

$\mathbb{Z}_N \times \mathbb{Z}_N$ action:

$$\begin{aligned} \mathbb{Z}_N \times \mathcal{A} &\to \mathcal{A} \\ (\omega^k, \phi) &\mapsto U^k \phi U^{-k} \end{aligned}$$

similar other $\mathbb{Z}_N$

$\mathcal{A} = \oplus_{n,m=0}^{N-1} U^n V^m$ ... harmonics

quantization map:

$$\begin{aligned} \mathcal{Q}: \quad \mathcal{C}(T^2) &\to \mathcal{A} = \mathrm{Mat}(\mathrm{N},\mathbb{C}) \\ e^{in\varphi}e^{im\psi} &\mapsto \begin{cases} q^{nm/2}U^nV^m, & |n|,|m| < N/2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

satisfies

$$\begin{aligned}\mathcal{Q}(fg) &= \mathcal{Q}(f)\mathcal{Q}(g) \; + O(\tfrac{1}{N}),\\ \mathcal{Q}(i\{f,g\}) &= [\mathcal{Q}(f),\mathcal{Q}(g)] \; + O(\tfrac{1}{N^2})\end{aligned}$$

Poisson structure $\{e^{i\varphi},e^{i\psi}\} = \frac{2\pi}{N}\,e^{i\varphi}e^{i\psi}$ on T^2 $\quad (\Leftrightarrow \{\varphi,\psi\} = -\frac{2\pi}{N})$

integral: $\quad 2\pi \mathrm{Tr}(\mathcal{Q}(f)) = \int\limits_{T^2} \omega_N f, \qquad \omega_N = \frac{N}{2\pi} d\varphi d\psi = N\omega_1$

T^2_N ... quantization of (T^2,ω_N)

metric on T^2_N ? ... "obvious", but need extra structure:

embedding $\quad T^2 \hookrightarrow \mathbb{R}^4$ via $\quad x^1 + ix^2 = e^{i\varphi},\; x^3 + ix^4 = e^{i\psi}$

quantization of embedding maps $x^a \sim X^a$: 4 hermitian matrices

$$X^1 + iX^2 := U, \qquad X^3 + iX^4 := V$$

satisfy

$$\begin{aligned}[][X^1,X^2] &= 0 = [X^3,X^4]\\ (X^1)^2+(X^2)^2 &= 1 = (X^3)^2+(X^4)^2\\ [U,V] &= (q-1)VU\end{aligned}$$

$\boxed{Exercise\ 4:}$ derive this, and translate the last relation into commutation relations for X^a

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Laplace operator:

$$\begin{aligned}\Box\phi &= [X^a,[X^b,\phi]]\delta_{ab}\\ &= [U,[U^\dagger,\phi]] + [V,[V^\dagger,\phi]] = 2\phi - U\phi U^\dagger - U^\dagger\phi U - (\%V)\\ \Box(U^nV^m) &= c([n]_q^2+[m]_q^2)\,U^nV^m \;\sim\; c(n^2+m^2)\,U^nV^m,\\ c &= -(q^{1/2}-q^{-1/2})^2 \sim \tfrac{1}{N^2}\end{aligned}$$

$\boxed{Exercise\ 5:}$ check this!

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where

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin(n\pi/N)}{\sin(\pi/N)} \sim n \qquad \text{("q-number")}$$

$$\boxed{\text{spec}\Box \approx \text{spec}\Delta_{T^2} \qquad \text{below cutoff}}$$

therefore:

geometry of (embedded) fuzzy torus $T_N^2 \hookrightarrow \mathbb{R}^4$ is $\approx$ that of a classical flat torus

momentum space is compactified! $[n]_q$

compare: noncommutative torus T_θ^2 Connes

$$\begin{aligned} UV &= qVU, \qquad q = e^{2\pi i\theta} \\ U^\dagger &= U^{-1}, \quad V^\dagger = V^{-1} \end{aligned}$$

generate $\mathcal{A}$ ... algebra of functions on T_θ^2

note: all $U^n V^m$ independent, $\mathcal{A}$ infinite-dimensional

in general non-integral (spectral) dimension, ...

for $\theta = \frac{p}{q} \in \mathbb{Q}$: ∞ -dim. center generated by $U^{nq} V^{mq}$

$$\text{fuzzy torus} \quad T_N^2 \quad \cong \quad T_\theta^2/\mathcal{C}, \qquad \theta = \frac{1}{N}$$

center $\mathcal{C}$ ... infinite additional sector (meaning ??)

NC torus T_θ^2 very subtle, "wild"

fuzzy torus T_N^2 "stable" under deformations

1.9 (Co)adjoint orbits

Let G ... compact Lie group with Lie algebra $\mathfrak{g} = Lie(G) \cong \mathbb{R}^D$.
Then G has a natural **adjoint action** on $\mathfrak{g}$ given by

$$g \triangleright X = Ad_g(X) = g \cdot X \cdot g^{-1}$$

for $g \in G$ and $X \in \mathfrak{g}$.

The (co-)adjoint orbit $\mathcal{O}[X]$ of G through $X \in \mathfrak{g}$ is then defined as

$$\mathcal{O}[X] := \{g \cdot X \cdot g^{-1} \,|\, g \in G\} \quad \subset \mathfrak{g} \cong \mathbb{R}^D$$

$\mathcal{O}[X]$ is submanifold embedded in "target space" $\mathbb{R}^D$, invariant under the adjoint action.
can assume that $X \in$ Cartan subalgebra, i.e. $X = H$ is diagonal.
is homogeneous space:

$$\mathcal{O}(H) \cong G/K_H$$

where $K_H = \{g \in G : \; Ad_g(H) = 0\}$ is the stabilizer of H.

choose ONB $\{\lambda_a, a = 1, ..., \dim \mathfrak{g}\}$ of $\mathfrak{g} \cong \mathbb{R}^D$,
structure constants

$$[\lambda_a, \lambda_b] = i f_{ab}^c \lambda_c$$

$\to$ Cartesian coordinate functions x^a on $\mathbb{R}^D \ni X = x^a \lambda_a$,
defines function

$$x^a : \quad \mathcal{O}[X] \quad \hookrightarrow \mathbb{R}^D$$

... characterize **embedding** of $\mathcal{O}[X]$ in $\mathbb{R}^D$, induce metric structure on $\mathcal{O}[X]$

1.9.1 Poisson structure on $\mathbb{R}^D$ and $\mathcal{O}[X]$:

$$\{x^a, x^b\} := f^{ab}_c x^c \tag{1}$$

extended to $\mathcal{C}^\infty(\mathbb{R}^D)$ as derivation.
Jacobi identity is consequence of Jacobi identity for $\mathfrak{g}$
adjoint action of $\mathfrak{g}$ on itself (=$\mathbb{R}^D$) is realized through Hamiltonian vector fields

$$ad_{\lambda_a}[X] = [\lambda_a, X] = -i\{x^a, X\}$$

Poisson structure is G- invariant
all Casimirs on $\mathfrak{g}$ are central, notably $C_2 \sim x_a x_b \, g^{ab}$
$\Rightarrow$ is not symplectic, but induces non-degenerate Poisson structure (symplectic structure) on $\mathcal{O}[X]$
the $\mathcal{O}[X]$ are the symplectic leaves of $\mathbb{R}^D$.

---

more abstract definition for symplectic structure:
G-invariant symplectic form on coadjoint orbit $\mathcal{O}^*_\mu$ ($\mu \in \mathfrak{g}$ ... weight)

$$\omega_\mu(\hat{X},\hat{Y}) := \mu([X,Y])$$

where $\hat{X}$ ... vector field on $\mathfrak{g}^*$ given by action of $X \in \mathfrak{g}$ on $\mathfrak{g}^*$.
... an antisymmetric, non-degenerate and closed 2-form on $\mathcal{O}^*_\mu$.
(Kirillov-Kostant-Souriau)

Example: sphere S^2_N
$G = SU(2)$, generators $\lambda_1, \lambda_2, \lambda_3$ = Pauli matrices
coadjoint orbit through

$$\lambda_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \in \mathfrak{su}(2)$$

stabilizer = $U(1)$

$$S^2 = \mathcal{O}[\lambda_3] \cong SU(2)/U(1)$$

Poisson bracket on $\mathbb{R}^3 = \mathfrak{su}(2)$

$$\{x_a, x_b\} = \epsilon_{abc}x_c$$

respects $R^2 = x_a x^a$, symplectic leaves = S^2.

Example: complex projective space $\mathbb{C}P^2$
$G = SU(3)$, generators λ_a = Gell-Mann matrices
coadjoint orbit through

$$\lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix} \qquad \in \mathfrak{su}(3)$$

stabilizer = $SU(2) \times U(1)$

$$\mathbb{C}P^2 = \mathcal{O}[\lambda_8] \cong SU(3)/SU(2) \times U(1)$$

Note:

$$X := 2\sqrt{3}\,\lambda_8 \qquad \text{satisfies} \qquad (X+1)(X-2) = 0$$

i.e. only two different eigenvalues

hence X determines a rank 1 projector

$$P := \frac{1}{3}(X + 1) \qquad \in Mat(3, \mathbb{C})$$

satisfies

$$P^2 = P, \qquad Tr(P) = 1$$

hence P can be written as

$$P = |z^i\rangle\langle z^i|$$

where $(z^i) = (z^1, z^2, z^3) \in \mathbb{C}^3$, normalized as $\langle z^i|z_i\rangle = 1$.
Such projectors are equivalent to rays in $\mathbb{C}^3$
$\rightarrow$ conventional description of $\mathbb{C}P^2$ as $\mathbb{C}^3/_{\mathbb{C}^*} \cong S^5/U(1)$.

Poisson bracket on $\mathbb{R}^8 = \mathfrak{su}(3)$

$$\{x_a, x_b\} = f_{abc}x_c$$

The **embedding** of $\mathcal{C}[X] \subset \mathbb{R}^8 \cong \mathfrak{su}(3)$ is described as follows:
characteristic equation $X^2 - X - 2 = 0$ is equivalent to

$$\delta_{ab}x^a x^b = 3, \qquad d^{abc}x^a x^b = \ x^c. \tag{2}$$

where d_{abc} is the totally symmetric invariant tensor of $SU(3)$.

$\boxed{Exercise\ 6}$: derive the relations (2) using $\lambda_a\lambda_b = \frac{2}{3}\delta_{ab} + \frac{1}{2}(if_{abc} + d_{abc})\lambda_c$

---

analogous construction for $\mathbb{C}P^n$:

$$\mathbb{C}P^n \cong \mathcal{O}(\lambda) \cong SU(n+1)/(SU(n) \times U(1))$$

is adjoint orbit of $SU(n+1)$ through maximally degenerate generator

$$\lambda \sim diag(-1, -1, ..., -1, n)$$

up to normalization.

1.9.2 Functions on $\mathcal{O}(\Lambda)$ & decomposition into harmonics:

G acts on $\mathcal{O}(\Lambda)$
$\rightarrow$ decompose classical algebra of polynomial functions on $\mathcal{O}(\Lambda)$:

$$Pol(\mathcal{O}(\Lambda)) = \oplus_\mu \, m_{\Lambda;\mu} V_\mu$$

where $m_{\mu;\Lambda} \in \mathbb{N}$ ... multiplicity

characterizes degrees of freedom on the space

1.10 Quantized coadjoint orbits embedded in $\mathbb{R}^D$

There is a canonical quantization for the above Poisson bracket on adjoint orbit with suitably quantized orbit.
Fact:
All finite-dimensional irreps V of G are given by highest weight representations, with dominant integral highest weight $\Lambda \in \mathfrak{g}_\circ{}^*$
Here $\mathfrak{g}_\circ \subset \mathfrak{g}$ is the Cartan subalgebra, i.e. max subalgebra of mutually commuting (i.e. diagonal) generators.
This means that $V = V_\Lambda$ has a unique highest weight vector $|\Lambda\rangle \in V$ with

$$\begin{aligned} X_i^+|\Lambda\rangle &= 0, \\ H|\Lambda\rangle &= H[\Lambda]\,|\Lambda\rangle \end{aligned}$$

for any (diagonal) Cartan generator H, and all other vectors in V are obtained by acting repeatedly with lowering operators X_i^- on $|\Lambda\rangle$.
(recall that the Lie algebra $\mathfrak{g}$ is generated by rising and lowering operators $X_i^\pm$ together with the Cartan generators.)

e.g. for $\mathfrak{su}(2)$: irreps characterized by spin, weights = eigevalue of $H = J_3$

Fact:
for compact Lie groups, there is a canonical isomorphism between the Lie algebra $\mathfrak{g}$ as a vector space and its dual space $\mathfrak{g}^*$, given by the standard Cartesian product $g_{ab} = \delta_{ab}$ on $\mathbb{R}^D$ (= Killing form).
In particular,

$$\Lambda \;\leftrightarrow H_\Lambda \tag{3}$$

Then coadjoint orbits $\mathcal{O}(\Lambda)$ through Λ are the same as adjoint orbits through H_Λ.

Given such a highest weight irrep $V_{N\Lambda}$, consider the matrix algebra

$$\mathcal{A}_N = End(V_{N\Lambda}) = Mat(\mathcal{N}), \qquad \mathcal{N} = \dim V_{N\Lambda}$$

G acts naturally on $\mathcal{A}_N$ via

$$\begin{aligned} G \times \mathcal{A}_N &\to \mathcal{A}_N \\ (g, M) &\mapsto \pi(g) M \pi(g^{-1}) \end{aligned} \tag{4}$$

where π ... rep. of G on $V_{N\Lambda}$ $\quad \to$ can decompose $\mathcal{A}$ into harmonics = irreps:

$$\mathcal{A}_N = End(V_{N\Lambda}) = V_{N\Lambda} \otimes V_{N\Lambda}^* = \oplus_\mu \tilde{m}_{N\Lambda;\mu} V_\mu$$

$\tilde{m}_{N\Lambda;\mu} \in \mathbb{N}$ ... multiplicity
can show:

$$\boxed{\tilde{m}_{N\Lambda;\mu} = m_{\Lambda;\mu}}$$

for sufficiently large N.
cf. (Hawkins q-alg/9708030, Pawelczyk & Steinacker hep-th/0203110)
moreover, can embed

$$\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1} ... \hookrightarrow Pol(\mathcal{O}(\Lambda))$$

preserving the group action and norms.

Hence: $\exists$ **quantization map**

$$\mathcal{Q}: \quad Pol(\mathcal{O}(\Lambda)) \to \mathcal{A}_N \tag{5}$$

$$Y^\mu_m \mapsto \begin{cases} \hat{Y}^\mu_m, & \mu < N \\ 0, & \mu \geq N \end{cases} \tag{6}$$

(schematically)
which respects the group action, the norm and is one-to-one for modes with sufficiently small degree μ.

"correspondence principle"

in practice: rescale as desired
In particular: **monomials = Lie algebra generators**

$$X^a := \mathcal{Q}(x^a) = c_N \pi(\lambda_a) \; = X^{a\dagger}$$

Their commutator reproduces Poisson bracket:

$$[X^a, X^b] = i c_N f^{abc} X^c \quad \stackrel{N\to\infty}{\to} \; 0 \tag{7}$$

$$\{x^a, x^b\} = c_N f^{abc} x^c \tag{8}$$

polynomial algebra generated by X^a generates full $\mathcal{A}_N = End(V_{N\Lambda})$.

Choose normalization e.g. such that

$$X^a X^a = c_N^2 \pi(\lambda_a \lambda^a) \stackrel{!}{=} R^2$$

here

$$\pi(\lambda_a \lambda^a) = C^2[N\Lambda] = (N\Lambda, N\Lambda + 2\rho) \sim N^2 \quad \text{...quadratic Casimir} \tag{9}$$

$$c_N \sim \frac{R}{N} \tag{10}$$

realize harmonics $\hat{Y}_m^\mu(X) \sim Y_m^\mu(x)$ e.g. as completely symmetric (traceless ...) polynomials of given degree.

Therefore:

Theorem 1.1. $\mathcal{A}_N = End(V_{N\Lambda})$ *provides a quantization* $\mathcal{O}_N(\Lambda)$ *of the coadjoint orbit* $\mathcal{O}(\Lambda)$*, viewed as Poisson (symplectic) manifold embedded in* $\mathbb{R}^D$ *with Poisson structure (8).*
same d.o.f. at low energies, but intrinsic UV cutoff.
The quantized embedding map is given by

$$X^a \;\propto\; \pi(\lambda^a)$$

The symplectic or Poisson structure is quantized such that

$$(2\pi)^n \mathrm{Tr}\, \mathbf{1} = \int \frac{\omega^n}{n!}$$

where $n = \dim \mathcal{O}(\Lambda)$

1.10.1 Example: fuzzy $\mathbb{C}P^2$

(Grosse & Strohmaier, Balachandran etal)

recall classical $\mathbb{C}P^2$:

$$\mathbb{C}P^2 \;\; = \;\; \{\lambda = g^{-1}\lambda_8 g, \quad g \in SU(3)\} \subset su(3) \cong \mathbb{R}^8 \qquad \text{... (co)adjoint orbit}$$

$\lambda = x^a \lambda_a$ satisfies embedding

$$\delta_{ab} x^a x^b = 3, \qquad d^{abc} x^a x^b = x^c. \tag{11}$$

harmonic analysis:

$$\mathcal{C}(\mathbb{C}P^2) \cong \oplus_{k=1}^{\infty} (k,k)$$

fuzzy version:

$$\mathcal{A}_N := \mathbb{C}P^2_N := End(V_N, \mathbb{C}) = \mathrm{Mat}(\mathrm{d_N}, \mathbb{C}) \cong \bigoplus_{\mathrm{k=1}}^{\mathrm{N}} (\mathrm{k}, \mathrm{k})$$

V_N ... irrep of $su(3)$ with highest weight $(0, N)$, $\quad d_N = \dim V_N = (N+1)(N+2)/2$

$$X^a = c_N \pi_N(\lambda_a), \qquad c_N = \frac{3}{\sqrt{N^2+3N}} \quad ,$$

is quantized embedding map

$$X^a \sim x^a : \mathbb{C}P^2 \hookrightarrow \mathbb{R}^8$$

can show: satisfies similar constraint

$$[X_a, X_b] = \frac{i}{\sqrt{N^2+3N}} f_{abc}\, X_c, \tag{12}$$

$$g_{ab}\, X_a X_b = 3, \tag{13}$$

$$d_{abc}\, X_a X_b = \frac{N+\frac{3}{2}}{\sqrt{N^2+3N}}\, X_c \tag{14}$$

reduces to (11) for $N \to \infty$,
Alexanian, Balachandran, Immirzi and Ydri hep-th/0103023, Grosse & Steinacker hep-th/0407089

1.11 Laplace operator on fuzzy $\mathcal{O}_N(X)$:

Let $\phi \in \mathcal{A}_N$ ... function on fuzzy $\mathcal{O}_N(X)$

Definition 1.3.

$$\Box\phi := g_{ab}[X^a, [X^b, \phi]]$$

where $X^a = \pi(\lambda_a) = X^{a\dagger}$ ... quantized embedding operators (possibly rescaled). Recall that $\mathfrak{g}$ acts via adjoint $J_a\phi := i[X_a, \phi]$ on $\mathcal{A}_N$

hence

$$\Box\phi = J_a J^a \phi$$
$$\Box\hat{Y}^\mu_m = C^2[\mu]\hat{Y}^\mu_m$$

quadratic Casimir
has same spectrum as classical Laplacian,

$$\Box_g Y^\mu_m \propto C^2[\mu] Y^\mu_m$$

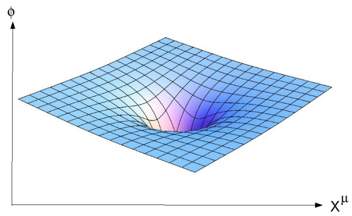
Thus $\Box$ has the same spectrum on $\mathcal{A}_N$ as $\Box_g$ on $\mathcal{C}^\infty(\mathcal{O}(\Lambda))$, up to cutoff.
hence:

$\Rightarrow$ $\mathcal{O}_N(\Lambda)$ has the same effective (spectral) geometry as $\mathcal{O}(\Lambda)$.

This is much more general, as we will see.

2 Generic fuzzy spaces

Framework is not restricted to homogeneous spaces.
General setup: D hermitian matrices $X^a \sim x^a: \quad \mathcal{M} \hookrightarrow \mathbb{R}^D$ describe quantized embedded symplectic space $(\mathcal{M}, \omega)$



inherits pull-back metric (geometry), (quantized) Poisson / symplectic structure is encoded via $[X^\mu, X^\nu] = i\theta^{\mu\nu}$

Define matrix Laplace operator on $\mathcal{M}_N$ by

$$\Box\phi := g_{ab}[X^a, [X^b, \phi]]$$

acting on $End(\mathcal{H})$
Similarly, let $\gamma_a, \; a = 1, ..., D$ ...Gamma matrices associated to $SO(D)$ acting on spinors V

$$\{\gamma_a, \gamma_b\} = 2g_{ab}$$

Define matrix Dirac operator by

$$\not{D} := \gamma_a \otimes [X^a, .].$$

acting on $V \otimes End(\mathcal{H})$.
Arises naturally in matrix models. Its square is given by

$$\not{D}^2 = \Box + \Sigma^{ab}[X^a, X^b]$$

where $\Sigma^{ab} := \frac{1}{4}[\gamma^a, \gamma^b]$.
(cf. Lichnerowicz formula)
$\boxed{Exercise\ 7:}$ check this relation.
These operators define a (spectral) geometry for $\mathcal{M}_N$.

2.1 Effective geometry of NC brane

consider scalar field moving on a fuzzy space, governed by "free" action

$$\begin{aligned} S[\varphi] &= -Tr\,[X^a, \varphi][X^b, \varphi]\, g_{ab} \\ &\sim \int \sqrt{|\theta^{-1}_{\mu\nu}|}\, \theta^{\mu'\mu} \partial_{\mu'} x^a \partial_\mu \varphi\, \theta^{\nu'\nu} \partial_{\nu'} x^b \partial_\nu \varphi\, g_{ab} \\ &= \int \sqrt{|G_{\mu\nu}|}\, G^{\mu\nu}(x)\, \partial_\mu \varphi \partial_\nu \varphi \end{aligned} \tag{15}$$

using $[f, \varphi] \sim i\theta^{\mu\nu}(x)\partial_\mu f \partial_\nu \varphi$
(assume $\dim \mathcal{M} = 4$)

$$\boxed{\begin{aligned} G^{\mu\nu}(x) &= e^{-\sigma} \theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x)\ g_{\mu'\nu'}(x) \quad \text{effective metric} \\ g_{\mu\nu}(x) &= \partial_\mu x^a \partial_\nu x^b g_{ab} \quad \text{induced metric on } \mathcal{M} \end{aligned}}$$

$$e^{-2\sigma} = \frac{|\theta^{-1}_{\mu\nu}|}{|g_{\mu\nu}|}$$

φ couples to metric $G^{\mu\nu}(x)$, determined by $\theta^{\mu\nu}(x)$ & embedding

... quantized Poisson manifold with metric $(\mathcal{M}, \theta^{\mu\nu}(x), G_{\mu\nu}(x))$

$\boxed{Exercise\ 8:}$ derive (15) with the above metric $G^{\mu\nu}$

2.1.1 The matrix Laplace operator

semi-classical limit of above matrix Laplacian:

Theorem 2.1. $(\mathcal{M},\omega)$ *symplectic manifold with* $\dim\mathcal{M}\neq 2$*, with* $x^a:\mathcal{M}\hookrightarrow\mathbb{R}^D$ *... embedding in* $\mathbb{R}^D$ *induced metric* $g_{\mu\nu}$ *and* $G^{\mu\nu}$ *as above. Then:*

$$\{x^a,\{x^b,\varphi\}\}g_{ab}=e^\sigma\Box_G\varphi$$

$\Box_G=\frac{1}{\sqrt{G}}\partial_\mu(\sqrt{G}G^{\mu\nu}\partial_\nu\phi)$ *... Laplace- Op. w.r.t.* $G_{\mu\nu}$
(H.S., [arXiv:1003.4134])

Hence: $\boxed{\Box\phi\sim -e^\sigma\Box_G\phi(x)}$

For coadjoint orbits: $G\sim g$ by group invariance, and $\Box\sim\Box_g$ follows.

2.2 A degenerate fuzzy space: Fuzzy S^4

H. Grosse, C. Klimcik and P. Presnajder, hep-th/9602115
(sketch; for more details see e.g.
Castelino, Lee & Taylor hep-th/9712105 or H.S. arXiv:1510.05779)

Classical construction:

Consider fundamental representation $\mathbb{C}^4$ of $SU(4)$. Acting on a reference point $z^{(0)}=(1,0,0,0)\in\mathbb{C}^4$, $SU(4)$ sweeps out the 7-sphere $S^7\subset\mathbb{R}^8\cong\mathbb{C}^4$
$\to$ Hopf map

$$S^7\to S^4\ \subset\mathbb{R}^5 \tag{16}$$
$$z^\alpha\mapsto x_i=z^*_\alpha(\gamma_i)^\alpha_\beta z^\beta\equiv\langle z|\gamma_i|z\rangle=tr(P_z\gamma_i),\quad P_z=|z\rangle\langle z| \tag{17}$$

where γ_i are the $\mathfrak{so}(5)$ gamma matrices.
Hence S^7 is a bundle over S^4 with fiber S^2.
Recall $\mathbb{C}P^3=S^7/U(1)$. Can quantize this! $\to$

Fuzzy construction:

Recall: $\mathfrak{su}(4)\cong\mathfrak{so}(6)$ generated by $\lambda^{ab}\in\mathfrak{so}(6)$
Start with fuzzy $\mathbb{C}P^3\subset\mathbb{R}^{15}\cong\mathfrak{su}(4)$, generated by

$$\mathcal{M}^{ab}=\pi_\mathcal{H}(\lambda^{ab})$$

acting on $\mathcal{H}_N=(0,0,N)$, for $1\leq a<b\leq 6$

Hopf map corresponds to composition

$$x^i:\ \ \mathbb{C}P^3\to\mathbb{R}^{15}\stackrel{\Pi}{\to}\mathbb{R}^5$$

where Π is projection of $\mathfrak{so}(6)$ to subspace spanned by $\lambda^{i6}, \ i = 1, ..., 5$

in other words: Fuzzy S^4_N is generated by

$$X^i := \mathcal{M}^{i6} \qquad End(\mathcal{H}) \tag{18}$$

for $\mathcal{H} = (0, 0, N)$ satisfy

$$\sum_{a=1}^{5} X^a X^a = \frac{1}{4} N(N+4)\mathbf{1}$$
$$[X_i, X_j] =: i\,\mathcal{M}_{ij}$$
$$[\mathcal{M}_{ij}, X_k] = i(\delta_{ik} X_j - \delta_{jk} X_i) \tag{19}$$

Is fully $SO(5)$-covariant fuzzy space, since $\mathcal{M}_{ij}, \ i, j = 1, ..., 5$ generate $\mathfrak{so}(5)$.

Snyder-type fuzzy space!

Can see: local fiber is fuzzy S^2_{N+1}.

2.3 A self-intersecting fuzzy space: squashed $\mathbb{C}P^2$

J. Zahn, H.S. : arXiv:1409.1440

classical construction:

Recall coadjoint orbit $\mathbb{C}P^2 \subset \mathbb{R}^8 \cong \mathfrak{su}(3)$

Consider projection map

$$\Pi : \mathbb{R}^8 \cong \mathfrak{su}(3) \to \mathbb{R}^6$$

projecting along the (simultaneously diagonalizable) Cartan generators λ^3, λ^8.

Then

$$x^a : \ \mathbb{C}P^2 \to \mathbb{R}^8 \stackrel{\Pi}{\to} \mathbb{R}^6$$

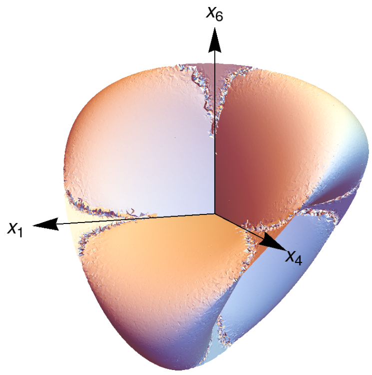
sefines a 4-dimensional subvariety of $\mathbb{R}^6$ with a triple self-intersection at the origin

Fuzzy construction:

generators

$$X^a = \pi_{\mathcal{H}}(\lambda^a), a = 1, 2, 4, 5, 6, 7 \qquad \in End(\mathcal{H})$$

acting on $\mathcal{H} = (0, N)$ generate fuzzy squashed $\Pi\mathbb{C}P^2_N$
arises as fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with soft SUSY breaking potential, and in an analogous modified IKKT matrix model
(3 generations etc.)



2.4 lessons

- algebra $\mathcal{A} = End(\mathcal{H})$ ... quantized algebra of functions on $(\mathcal{M}, \omega)$
  no geometrical information (not even $\dim$)
  $\dim(\mathcal{H})$ = number of "quantum cells", $(2\pi)^n \mathrm{Tr}\, \mathcal{Q}(f) \sim \mathrm{Vol}_\omega \mathcal{M}$
- Poisson/symplectic structure encoded in C.R.
- every non-deg. fuzzy space locally $\approx \mathbb{R}^{2n}_\theta$ (cf. Darboux theorem!)
- geometrical info encoded in *specific matrices* $X^a, \ a = 1, ..., D$:

  $$X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D \qquad \text{...embedding functions}$$

  contained e.g. in matrix Laplacian $\Box = [X^a, [X^b, .]]\delta_{ab}$
  (or in coherent states, see below).

3 Lecture II: Applications: NC field theory & matrix models

goal: formulate physical models on fuzzy spaces
scalar field theory, gauge theory, ("emergent") gravity
issues:
quantization $\Rightarrow$ UV/IR mixing due to $\Delta x^\mu \Delta x^\nu \geq L_{NC}^2$
$\rightarrow$ strong **non-locality**, can be traced to **string states**
$\rightarrow$ selects well-behaved model: IKKT model =maximal SUSY matrix model

(Yang-Mills) Matrix Models, IKKT model

- describes dynamical fuzzy branes = submanifolds $\mathcal{M} \hookrightarrow \mathbb{R}^{10}$
  interpreted as physical space-time
- NC gauge theory, dynamical geometry & emergent gravity
- closely related to string theory, introduced as non-perturbative description of string theory on $\mathbb{R}^{10}$
- well-behaved under quantization, due to maximal SUSY

3.1 Scalar field theory on S_N^2

consider $\mathcal{A}_N = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... (Hilbert) space of functions on S_N^2

action for free real scalar field $\phi = \phi^\dagger$:

$$
\begin{aligned}
S_0[\phi] &= \tfrac{4\pi}{N}\mathrm{Tr}(\tfrac{1}{2}\phi\Box\phi + \tfrac{1}{2}\mu^2\phi^2) \\
&\sim \int_{S^2}(\tfrac{1}{2c_N}\phi\Delta_g\phi + \tfrac{1}{2}\mu^2\phi^2)
\end{aligned}
$$

harmonic ("Fourier") decomposition

$$
\phi = \sum_{lm} \phi_{lm}\hat{Y}^l_m, \qquad \phi^\dagger_{l,m} = \phi_{l,-m}
$$

(finite!)

$$
S_0[\phi] = \tfrac{4\pi}{N}\sum_{l,m}(\phi_{l,m}(l(l+1)+\mu^2)\phi_{l,m})
$$

interacting real scalar field:

$$
\begin{aligned}
S[\phi] &= \tfrac{4\pi}{N}\mathrm{Tr}(\tfrac{1}{2}\phi\Box\phi + \tfrac{1}{2}\mu^2\phi^2 + \lambda\phi^4) \\
&= \sum \phi_{l,m}\phi_{l,-m}(l(l+1)+\mu^2) + \lambda\sum \phi_{l_1 m_1}\cdots\phi_{l_n m_n}V_{l_1..l_4;m_1...m_4}, \\
V_{l_1..l_4;m_1...m_4} &= \mathrm{Tr}(\hat{Y}^{l_1}_{m_1}...\hat{Y}^{l_4}_{m_4})
\end{aligned}
$$

... deformation of classical FT on S^2, built-in UV cutoff

3.2 scalar QFT on S^2_N

Feynman "path" (matrix) integral approach

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi e^{-S[\phi]+Tr\phi J} \\
\langle \phi_{l_1 m_1} \cdots \phi_{l_n m_n} \rangle &= \frac{\int [\mathcal{D}\phi] e^{-S[\phi]}\ \phi_{l_1 m_1} \cdots \phi_{l_n m_n}}{\int [\mathcal{D}\phi] e^{-S[\phi]}} \\
&= \frac{1}{Z[0]} \frac{\partial^n}{\partial J_1 \ldots \partial J_n} Z[J]|_{J=0}, \qquad [\mathcal{D}\phi] = \prod d\phi_{lm} \\
& \qquad\qquad\qquad\qquad \phi = \sum \phi_{lm} \hat{Y}^l_m
\end{aligned}
$$

... deformation & regularization of (euclid.) QFT on S^2,
finite version of path integral,
UV cutoff

<u>free QFT:</u>

$$
S_0[\phi] =: Tr \frac{1}{2} \phi D \phi = \frac{1}{2} \sum \phi_{l_1 -m_1} (l(l+1) + \mu^2) \phi_{l_1 m_1} \tag{20}
$$

Gaussian integral,

$$
\begin{aligned}
Z[J] &= \int d\phi e^{-Tr(\frac{1}{2}\phi D \phi - \phi J)} = \int d\phi e^{-Tr(\frac{1}{2}(\phi - D^{-1}J)^\dagger D (\phi - D^{-1}J) + \frac{1}{2} Tr J D^{-1} J} \\
&= \frac{1}{\mathcal{N}} e^{\frac{1}{2} Tr J D^{-1} J}
\end{aligned} \tag{21}
$$

propagator:

$$
\begin{aligned}
\langle \phi_{l_1 m_1} \phi_{l_2 m_2} \rangle &= \frac{1}{Z} \int \prod d\phi_{lm}\ \phi_{l_1 m_1} \phi_{l_2 m_2} e^{-\sum \phi_{l,m}\phi_{l,-m}(l(l+1)+\mu^2)} \\
&= \frac{1}{Z[0]} \frac{\partial^2}{\partial J_1 \partial J_2} Z[J]|_{J=0} \\
&= \delta_{l_1 l_2} \delta_{m_1,-m_2} \frac{1}{l(l+1)+\mu^2}
\end{aligned}
$$

as in commutative case, up to cutoff

$\rightarrow$ free field theory coincides with undeformed one.

<u>interacting QFT:</u>

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi e^{-S_0[\phi] - S_{int}[\phi] + Tr\phi J} \\
&= e^{-S_{int}[\partial_J]} \int \mathcal{D}\phi e^{-S_0[\phi] + Tr\phi J} \\
&= e^{-S_{int}[\partial_J]} Z[J] \\
\langle \phi_{I_1} \ldots \phi_{I_{2n}} \rangle &= \frac{1}{Z[0]} \frac{\partial^n}{\partial J_1 \ldots \partial J_n} e^{-S_{int}[\partial_J]} Z[J]|_{J=0}
\end{aligned}
$$

perturbative expansion $\Rightarrow$ Wick's theorem,

$$\langle \phi_{I_1} ... \phi_{I_{2n}} \rangle = \sum_{\text{contractions}} \langle \phi\phi \rangle ... \langle \phi\phi \rangle$$

vertices: e.g.

$$\begin{aligned} S_{int} &= \tfrac{1}{4!} Tr\phi^4 = TrV, \\ V &= \lambda \sum \phi_{l_1 m_1} \cdots \phi_{l_n m_n} V_{l_1 .. l_4; m_1 ... m_4} \end{aligned}$$

finite, but distinction planar $\leftrightarrow$ nonplanar diagrams

results: hep-th/0106205

- large phase factors, interaction vertices rapidly oscillating for $\frac{l_1}{R}\frac{l_2}{R} \geq \Lambda_{NC}^2$

  (loop effects probe area quantum $\Delta A \sim 1/N$)

- 1-loop effective action

$$S_{one-loop} = S_0 + \int \frac{1}{2} \, \Phi(\delta\mu^2 - \frac{g}{12\pi} h(\widetilde{\Delta}))\Phi + o(1/N)$$

with

$$h(l) = -\frac{1}{2} \int_{-1}^{1} dt \frac{1}{1-t}(P_l(t) - 1) = (\sum_{k=1}^{l} \frac{1}{k}), \qquad \delta\mu^2 = \frac{g}{8\pi} \sum_{j=0}^{N} \frac{2j+1}{j(j+1)+\mu^2}$$

Chu Madore HS [hep-th/0106205]

does NOT agree with usual QFT on S^2,

"anomalous contributions" to quantum effective action

(=finite version of UV/IR mixing)

central feature of NC QFT, obstacle for perturb. renormalization

Minwalla, V. Raamsdonk, Seiberg hep-th/9912072]

- new physics, **non-local**

4 Matrix models and NC gauge theory

4.1 Matrix model for S^2_N

$$\begin{aligned} S[X] &= \frac{1}{g^2}\mathrm{Tr}\left([X^a,X^b][X_a,X_b] - 4i\varepsilon_{abc}X^aX^bX^c - 2X^aX_a\right) \\ &= \frac{1}{g^2}\mathrm{Tr}\left(([X^a,X^b] - i\varepsilon^{abc}X_c)([X_a,X_b] - i\varepsilon_{abc}X^c)\right) \\ &= \frac{1}{g^2}\mathrm{Tr}\, F^{ab}F_{ab} \qquad \geq 0 \end{aligned}$$

where $X^a \in \mathrm{Mat}(\mathrm{N},\mathbb{C}), \quad \mathrm{a} = 1,2,3$ and

$$F^{ab} \; := \; [X^a,X^b] - i\varepsilon^{abc}X_c \qquad \text{field strength}$$

solutions (minima!):

$$\begin{aligned} F^{ab} &= 0 \quad \Leftrightarrow \quad [X^a,X^b] = i\varepsilon^{abc}X_c \\ X^a &= \lambda^a, \qquad \lambda^a \ldots \text{rep. of } \mathfrak{su}(2) \end{aligned}$$

any rep. of $su(2)$ is a solution! $X^a = \begin{pmatrix} \lambda^a_{(M_1)} & 0 & \ldots & 0 \\ 0 & \lambda^a_{(M_2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda^a_{(M_k)} \end{pmatrix}$

concentric fuzzy spheres $S^2_{M_i}$!
geometry & topology dynamical !
expand around solution:

$$X^a = \lambda^a + A^a \qquad \in \mathrm{Mat}(\mathrm{N},\mathbb{C})$$

$$F^{ab} \; = \; [\lambda^a, A^b] - [\lambda^b, A^a] - i\varepsilon^{abc}A_c + [A^a,A^b] \qquad (\sim \text{“}dA + AA\text{''})$$

can be interpreted in terms of

- $U(1)$ gauge theory on S^2_N (tang. fluct. if) $\lambda^a A_a = 0$
- coupled to scalar field $D_\mu\phi D^\mu\phi$ (radial fluctuations) $X^a = \lambda^a(1+\phi)$

however:
radial deformation = deformation of embedding, geometry!

geometry $\leftrightarrow$ NC gauge theory ?!

above matrix model describes dynamical fuzzy space

4.2 Gauge theory on $\mathbb{R}^4_\theta$

Let $[\bar X^\mu, \bar X^\nu] = i\bar\theta^{\mu\nu}, \qquad \bar X^\mu \in \mathcal{A} = \mathcal{L}(\mathcal{H})$ (Moyal-Weyl) consider fluctuations around $\mathbb{R}^4_\theta$:

$$X^\mu = \bar X^\mu - \bar\theta^{\mu\nu} A_\nu, \qquad A_\nu \in \mathcal{A}$$

Define derivative operator on $\mathbb{R}^4_\theta$ by $[\bar X^\mu, \phi] = i\theta^{\mu\nu}\partial_\nu\phi \ \rightarrow$

$$\begin{aligned} [X^\mu, X^\nu] &= i\bar\theta^{\mu\nu} - i\bar\theta^{\mu\mu'}\bar\theta^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'} + i[A_{\mu'}, A_{\nu'}]) \\ &= i\bar\theta^{\mu\nu} - i\bar\theta^{\mu\mu'}\bar\theta^{\nu\nu'} F_{\mu'\nu'} \end{aligned}$$

$F_{\mu\nu}(x)$ ... $\mathfrak{u}(1)$ field strength

Exercise 12: check this formula (and the gauge transformation law below)

Yang-Mills action:

$$\begin{aligned} S_{YM}[X] &= \mathrm{Tr}[X^\mu, X^\nu][X^{\mu'}, X^{\nu'}]\delta_{\mu\mu'}\delta_{\nu\nu'} \\ &= \rho \int d^4x F_{\mu\nu}F_{\mu'\nu'}\bar G^{\mu\mu'}\bar G^{\nu\nu'} \quad + \partial() \end{aligned}$$

(up to surface term $\mathrm{Tr}[X, X] = \int F \ \rightarrow 0$)

... NC $U(1)$ gauge theory on $\mathbb{R}^4_\theta$,

effective metric

$$\bar G^{\mu\nu} = \bar\theta^{\mu\mu'}\bar\theta^{\nu\nu'}\delta_{\mu'\nu'}, \qquad\qquad \rho = |\bar\theta^{-1}_{\mu\nu}|^{1/2}$$

gauge transformations:

$$\begin{aligned} X^\mu \rightarrow UX^\mu U^{-1} = \ & U(\bar X^\mu - \bar\theta^{\mu\nu} A_\nu)U^{-1} = \bar X^\mu + U[\bar X^\mu, U^{-1}] - \bar\theta^{\mu\nu}UA_\nu U^{-1} \\ = \ & \bar X^\mu - \bar\theta^{\mu\nu}(U\partial_\nu U^{-1} + UA_\nu U^{-1}) \end{aligned}$$

infinitesimal $\quad U = e^{i\Lambda(X)}, \quad \delta A_\mu = i\partial_\mu\Lambda(X) + i[\Lambda(X), A_\mu]$

invariant under gauge trafo

$$F_{\mu\nu} \ \rightarrow \ UF_{\mu\nu}U^{-1} \quad \sim \quad \text{symplectomorphism}$$

Yang-Mills matrix model S_{YM} describes $U(1)$ gauge theory on $\mathbb{R}^4_\theta$

no "local" observables ! (need trace)

coupling to scalar fields:

consider

$$\begin{aligned} S[X,\phi^i] &= -\mathrm{Tr}\left([X^\mu,X^\nu][X^{\mu'},X^{\nu'}]\delta_{\mu\mu'}\delta_{\nu\nu'} + [X^\mu,\phi][X^{\mu'},\phi]\delta_{\mu\mu'}\right) \\ &= \rho\int d^4x\left(F_{\mu\nu}F_{\mu'\nu'}\,\bar G^{\mu\mu'}\bar G^{\nu\nu'} + D_\mu\phi D_\nu\phi\,\bar G^{\mu\nu}\right) \end{aligned}$$

$$[X^\mu,\phi] = i\bar\theta^{\mu\nu}(\partial_\nu + i[A_\mu,.])\phi =: i\bar\theta^{\mu\nu}D_\mu\phi$$

gauge transformation

$$\phi \to U\phi U^{-1} \qquad \text{(adjoint)}$$

same form as

$$S[X] = \mathrm{Tr}[X^a,X^b][X^{a'},X^{b'}]\delta_{aa'}\delta_{bb'}, \qquad a = 1,...,4+1$$

more generally: $D = 10$ matrix model around $\mathbb{R}^4_\theta$:

$$\begin{aligned} S[X] &= -\mathrm{Tr}[X^a,X^b][X^{a'},X^{b'}]\delta_{aa'}\delta_{bb'} = \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^4_\theta} d^4x\,\rho\Big(\bar G^{\mu\mu'}\bar G^{\nu\nu'}\,F_{\mu\nu}\,F_{\mu'\nu'} + \bar G^{\mu\nu}\eta_{\mu\nu} \\ &\quad +2\,\bar G^{\mu\nu}\,D_\mu\phi^i D_\nu\phi^j\delta_{ij} + [\phi^i,\phi^j][\phi^{i'},\phi^{j'}]\delta_{ii'}\delta_{jj'}\Big) \end{aligned}$$

... same as bosonic part of $\mathcal{N}=4$ SYM!

generalization to $U(n)$:

new background

$$X^a \otimes \mathbf{1}_n$$

naturally interpreted as n coincident branes. fluctuations

$$\begin{pmatrix} X^\mu \\ \phi^i \end{pmatrix} = \begin{pmatrix} \bar X^\mu \otimes \mathbf{1}_n \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{A}^\mu \\ \phi^i \end{pmatrix},$$

it is easy to see that

$$\mathcal{A}^\mu = -\theta^{\mu\nu}A_{\nu,\alpha}(\bar X)\lambda^\alpha, \qquad \phi^i = \phi^i_\alpha(\bar X)\lambda^a$$

... $\mathfrak{u}(n)$- valued gauge resp. scalar fields on $\mathbb{R}^4_\theta$, denoting with λ^α a basis of $\mathfrak{u}(n)$.

The matrix model

$$\begin{aligned} S &= -\mathrm{Tr}[X^a,X^b][X^{a'},X^{b'}]\delta_{aa'}\delta_{bb'} = \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^4_\theta} d^4x\,\rho tr\Big(\bar G^{\mu\mu'}\bar G^{\nu\nu'}\,F_{\mu\nu}\,F_{\mu'\nu'} + \bar G^{\mu\nu}\eta_{\mu\nu}\mathbf{1}_n \\ &\quad +2\,\bar G^{\mu\nu}\,D_\mu\phi^i D_\nu\phi^j\delta_{ij} + [\phi^i,\phi^j][\phi^{i'},\phi^{j'}]\delta_{ii'}\delta_{jj'}\Big) \end{aligned}$$

where $tr()$ ... trace over the $\mathfrak{u}(n)$ matrices,
$F_{\mu\nu}$ ... $\mathfrak{u}(n)$ field strength.
... $\mathfrak{u}(n)$ Yang-Mills on $\mathbb{R}^4_\theta$
<u>note:</u>

- extremely simple mechanism:

  gauge fields = fluctuations of dynamical matrices $X^\mu \to X^\mu + \mathcal{A}^\mu$

  "covariant coordinates"

  works only on NC spaces!

- matrix models $\mathrm{Tr}[X, X][X, X]$ $\sim$ gauge-invariant YM action

- generalized easily to $U(n)$ theories but

  $U(1)$ sector does not decouple from $SU(n)$ sector

- one-loop: UV/IR mixing $\to$ not QED, problem

  except in $\mathcal{N} = 4$ SUSY case: finite (!?)

- closer inspection: $U(1)$ sector is part of geometric sector,

  $\to$ emergent "gravity".