Lecture series on 3d gravity

Lecture 3: Quantisation

Quantum Structure of Spacetime and Gravity 2016

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EUROPEAN COOPERATION
IN SCIENCE AND TECHNOLOGY

Quantisation approaches to 3d gravity

discrete path integral methods

- **state sum models** of 3d gravity, aka **spin foams**
- based on 3d manifold + discretisation
- context: **3d topological quantum field theory**
- context: **3-manifold invariants**

- **Ex: Reshetikhin-Turaev invariant, Turaev-Viro invariant**

⇨ based on of Hopf algebras, (quantum groups) and their representation theory

Hamiltonian quantisation approaches

- based on surface S for 3-manifold $\mathbb{R} \times S$
- quantisation of phase space, canonical quantisation
- context: **quantisation of moduli spaces of flat connections**

- **Ex: combinatorial quantisation of Chern-Simons theory**

⇨ based on of Hopf algebras, (quantum groups) and their representation theory

from quantisation of Teichmüller space

- in principle promising, but no complete picture yet

⇨ **focus on Hamiltonian quantisation approaches, combinatorial quantisation**

⇨ **focus on mathematical structures, in simplified framework**

1. Hopf algebras

= associative unital algebras with representation theory like the one of a group

group G	**associative algebra K over $\mathbb{F}$**
representation on V	**representation on V**
group homomorphism $\rho_V : G \to \mathrm{Aut}_{\mathbb{F}}(V)$	algebra homomorphism $\rho_V : K \to \mathrm{End}_{\mathbb{F}}(V)$
• **representations on tensor products:** $\rho_{V\otimes W} : G \to \mathrm{Aut}_{\mathbb{F}}(V\otimes W)$ $\rho_{V\otimes W}(g)(v\otimes w) = \rho_V(g)v \otimes \rho_W(g)w$	• **algebra homomorphism** $\Delta : K \to K\otimes K$ **comultiplication** $\rho_{V\otimes W} : K \to \mathrm{End}_{\mathbb{F}}(V\otimes W)$ $\rho_{V\otimes W} = (\rho_V \otimes \rho_W)\circ\Delta$
• **trivial representation:** $\rho_{\mathbb{F}} : G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}) \quad \rho_{\mathbb{F}}(g) = \mathrm{id}_{\mathbb{F}}$	• **algebra homomorphism** $\rho_{\mathbb{F}} = \epsilon : K \to \mathbb{F}$ **counit**
• **representation on dual vector space:** $\rho_{V^*} : G \to \mathrm{Aut}_{\mathbb{F}}(V^*) \quad \rho_{V^*}(g)\alpha = \alpha\circ\rho_V(g^{-1})$	• **anti-algebra homomorphism** $S : K \to K$ **antipode** $\rho_{V^*} = \rho_V \circ S$

consistency conditions:

$U\otimes(V\otimes W)\cong(U\otimes V)\otimes W$ ⟶ **coassociativity** $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$

$\mathbb{F}\otimes V\cong V\cong V\otimes\mathbb{F}$ ⟶ **counitality** $(\epsilon\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\epsilon)\circ\Delta = \mathrm{id}$

$V\otimes V^*\to\mathbb{F}\leftarrow V^*\otimes V$ ⟶ **condition** $m\circ(S\otimes\mathrm{id})\circ\Delta = m\circ(\mathrm{id}\otimes S)\circ\Delta = \epsilon\,1$

Def A **Hopf algebra** is an associative algebra $(K, m, 1)$ over $\mathbb{F}$ with algebra homomorphisms $\Delta : K \to K \otimes K$ and $\epsilon : K \to \mathbb{F}$ and an anti algebra homomorphism $S : K \to K$ that satisfy

coassociativity	$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$
counitality	$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$
antipode condition	$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \epsilon\, 1$

cocommutative $\Delta^{op} = \Delta$

Sweedler notation $\Delta(k) = k_{(1)} \otimes k_{(2)}$

- **Ex:** group algebra of finite group $\mathbb{F}[G]$

  $\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad S(g) = g^{-1}$

  cocommutative $\Delta^{op} = \Delta$

  antipode is involution $S^2 = \mathrm{id}$

- **Ex:** universal enveloping algebra of a Lie algebra

  $U(\mathfrak{g}) = T(\mathfrak{g})/(\mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x} - [\mathbf{x}, \mathbf{y}])$

  $\Delta(\mathbf{x}) = \mathbf{x} \otimes 1 + 1 \otimes \mathbf{x} \qquad \epsilon(\mathbf{x}) = 0 \qquad S(\mathbf{x}) = -\mathbf{x}$

  cocommutative $\Delta^{op} = \Delta$

  antipode is involution $S^2 = \mathrm{id}$

- **fact:** K finite-dim and $\mathrm{char}(\mathbb{F}) = 0$: K semisimple $\Leftrightarrow$ $S^2 = \mathrm{id}$
- **fact:** K finite-dim, semisimple, cocommutative, $\mathbb{F}$ alg closed with $\mathrm{char}(\mathbb{F}) = 0 \Rightarrow K = \mathbb{F}[G]$
- **fact:** K finite-dim, semisimple $\Rightarrow$ K has normalised Haar integral

  $\ell \in K$ with $\epsilon(\ell) = 1$ and $k \cdot \ell = \ell \cdot k = \epsilon(k)\, \ell \;\; \forall k \in K$

- **Ex:** Hopf algebra structure on K $\Leftrightarrow$ Hopf algebra structure on K^*

$$\langle\ ,\ \rangle : K^* \otimes K \to \mathbb{F} \qquad \alpha \otimes k \mapsto \alpha(k)$$

$$\langle \alpha \cdot \beta, k \rangle = \langle \alpha \otimes \beta, \Delta(k) \rangle \qquad \langle 1, k \rangle = \epsilon(k) \qquad \langle S(\alpha), k \rangle = \langle \alpha, S(k) \rangle$$

$$\langle \Delta(\alpha), h \otimes k \rangle = \langle \alpha, h \cdot k \rangle \qquad \epsilon(\alpha) = \langle \alpha, 1 \rangle$$

- **Ex:** $K = \mathbb{F}[G] \ \Rightarrow K^* = \mathrm{Fun}(G)$

$$\langle f, g \rangle = f(g) \qquad f_1 \cdot f_2(g) = f_1(g) \cdot f_2(g) \qquad 1(g) = 1 \qquad S(f)(g) = f(g^{-1})$$

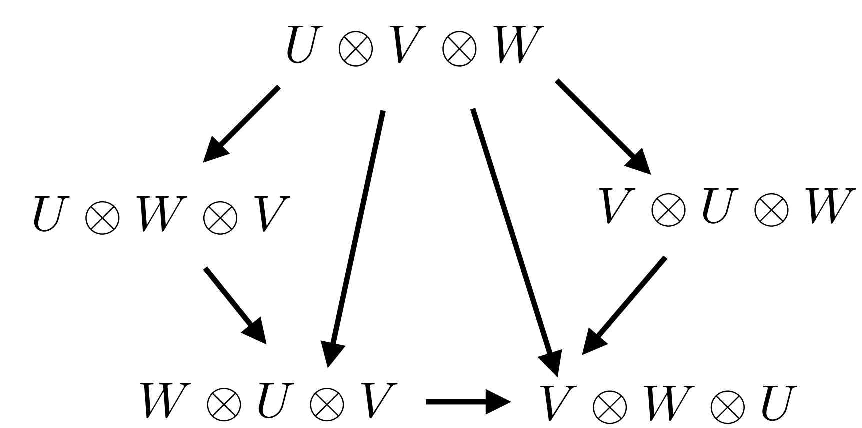
$$\Delta(f)(g_1, g_2) = f(g_1 g_2) \qquad \epsilon(f) = f(1)$$

Def A Hopf algebra K is **quasitriangular** if there is an element $R \in K \otimes K$ with $\Delta^{op} = R \cdot \Delta \cdot R^{-1}$ and $(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23}$ $\quad (\mathrm{id} \otimes \Delta)(R) = R_{13} R_{12}$

⇨ **representation theoretical consequence:**

module map $\phi_{U,V} : U \otimes V \to V \otimes U$

$$\phi_{U,V}(u \otimes v) = \tau \circ (\rho_U \otimes \rho_V)(R^{-1})(u \otimes v)$$



- **Ex: Drinfeld double $D(H)$ of a finite-dim Hopf algebra H**

$$D(H) = H^* \otimes H$$

$$(\alpha \otimes h) \cdot (\beta \otimes k) = \langle h_{(1)}, \beta_{(3)} \rangle \langle h_{(3)}, S^{-1}(\beta_{(1)}) \rangle\, \alpha \beta_{(2)} \otimes h_{(2)} k$$

$$\Delta(\alpha \otimes h) = \alpha_{(2)} \otimes h_{(1)} \otimes \alpha_{(1)} \otimes h_{(2)}$$

$$S(\alpha \otimes h) = (1 \otimes S(h)) \cdot (S^{-1}(\alpha) \otimes 1)$$

$\Rightarrow$ quasitriangular with

$$R = \Sigma_i\, 1 \otimes x_i \otimes \alpha^i \otimes 1$$

2. Module algebras over Hopf algebras

- Poisson-Lie group G ⇨ algebra of functions $C^\infty(M)$ on Poisson G -space M
- Hopf algebra K ⇨ module algebra A over K

Def A **left module algebra** A over a Hopf algebra K is an associative algebra A with a K-left module structure $\triangleright : K \otimes A \to A$ such that

$$k \triangleright (a \cdot b) = (k_{(1)} \triangleright a) \cdot (k_{(2)} \triangleright b) \qquad k \triangleright 1 = \epsilon(k)\, 1$$

similarly: **right module algebra** and **bimodule algebra** over K

⇨ **invariants** $A_{inv} = \{a \in A : \; k \triangleright a = \epsilon(k)\, a \, \forall k \in K\}$

~ invariant functions $C^\infty(M)^G$ for Poisson G - space M

- **fact:**

  - A left module algebra over K $\Rightarrow$ $A_{inv} = \{a \in A : \; k \triangleright a = \epsilon(k)\, a \, \forall k \in K\}$ subalgebra
  - K semisimple $\Rightarrow$ projector $P : A \to A$, $a \mapsto \ell \triangleright a$ on A_{inv} from Haar integral $\ell \in K$

- **Ex: group action on set** $\triangleright : G \times X \to X$

$K = \mathbb{F}[G]$ $\Rightarrow$ K- right module algebra structure on A $\quad \triangleleft : \mathrm{Fun}(X) \otimes \mathbb{F}[G] \to \mathrm{Fun}(X)$

$A = \mathrm{Fun}(X)$ $\quad (f \triangleleft g)(x) = f(g \triangleright x)$

- **Ex: adjoint action** $\triangleright : K \otimes K \to K$ $\Rightarrow$ K- left module algebra structure on K
  ~ conjugation $k \triangleright h = k_{(1)} \cdot h \cdot S(k_{(2)})$

- **Ex: left regular action** $\triangleleft : K^* \otimes K \to K^*$
  dual to left multiplication $\alpha \triangleleft k = \langle \alpha_{(1)}, k \rangle \, \alpha_{(2)}$

- **Ex: right regular action** $\triangleright : K \otimes K^* \to K^*$
  dual to right multiplication $k \triangleright \alpha = \langle k, \alpha_{(2)} \rangle \, \alpha_{(1)}$

$\Rightarrow$ K- right module algebra structure on K^*

$\Rightarrow$ K- bimodule algebra structure on K^*

$\Rightarrow$ K- left module algebra structure on K^*

- **fact:**

A right module algebra over K

$\Rightarrow$ algebra structure on $A \otimes K$: $(a \otimes k) \cdot (b \otimes h) = a(b \triangleleft k_{(2)}) \otimes k_{(1)} h$ cross product $K \# A$

- **Ex: Heisenberg double**

cross product for left regular action $\triangleleft : K^* \otimes K \to K^*$

$K \otimes K^*$ with algebra structure $(h \otimes \alpha) \cdot (k \otimes \beta) = \langle \alpha_{(1)}, k_{(2)} \rangle \, h k_{(1)} \otimes \alpha_{(2)} \beta$

3. Hopf algebra gauge theory

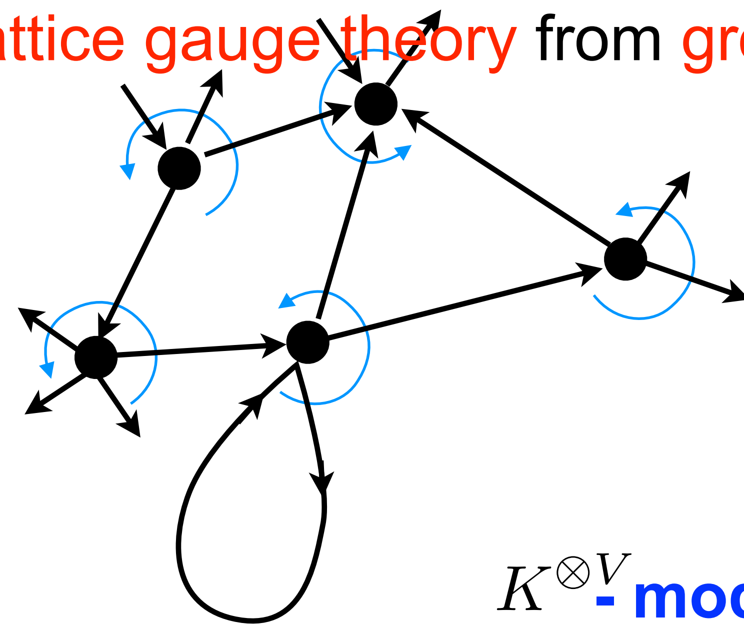
mathematical structures for lattice gauge theory with values in a Hopf algebra

generalise concept of lattice gauge theory from group to Hopf algebra

lattice gauge theory for a group G

ribbon graph Γ

lattice gauge theory for a finite-dim Hopf algebra K



gauge fields **set** $G^{\times E}$

$\updownarrow$ evaluation $(f,g)\mapsto f(g)$

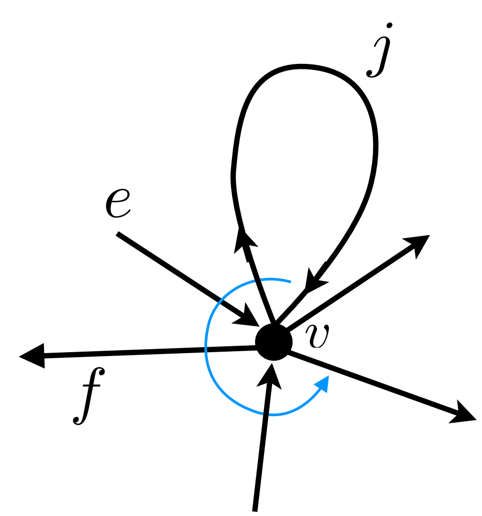
functions **algebra** $\mathrm{Fun}(G^{\times E})$

vector space $K^{\otimes E}$

$\updownarrow$ pairing $\langle\ ,\ \rangle : K^*\otimes K\to\mathbb{F}$

$K^{\otimes V}$**-module** **algebra structure** on $K^{*\otimes E}$

$$(\alpha\cdot\beta)\lhd h=(\alpha\lhd h_{(1)})\cdot(\beta\lhd h_{(2)})$$
$$1\lhd h=\epsilon(h)\,1$$



gauge transformations **group** $G^{\times V}$

action on connections and functions

$$\lhd:\mathrm{Fun}(G^{\times E})\times G^{\times V}\to\mathrm{Fun}(G^{\times E})$$
$$(f\lhd h)(g)=f(h\rhd g)$$

$$g_e\mapsto g_v\cdot g_e$$
$$g_f\mapsto g_f\cdot g_v^{-1}$$
$$g_j\mapsto g_v\cdot g_j\cdot g_v^{-1}$$

Hopf algebra $K^{\otimes V}$

module structures

$$\lhd: K^{*\otimes E}\otimes K^{\otimes V}\to K^{*\otimes E}$$
$$\langle\alpha\lhd h,k\rangle=\langle\alpha,h\rhd k\rangle$$

observables $f\in\mathrm{Fun}(G^{\times E})$

$f\lhd h=f\quad\forall h\in G$

subalgebra of $\mathrm{Fun}(G^{\times E})$

left regular action
right regular action
adjoint action

invariants $\alpha\in K^{*\otimes E}$

$\alpha\lhd h=\epsilon(h)\,\alpha\quad\forall h\in K^{\otimes V}$

subalgebra of $K^{*\otimes E}$

construction of Hopf algebra gauge theory

setting: K finite-dim semisimple Hopf algebra, Γ ribbon graph

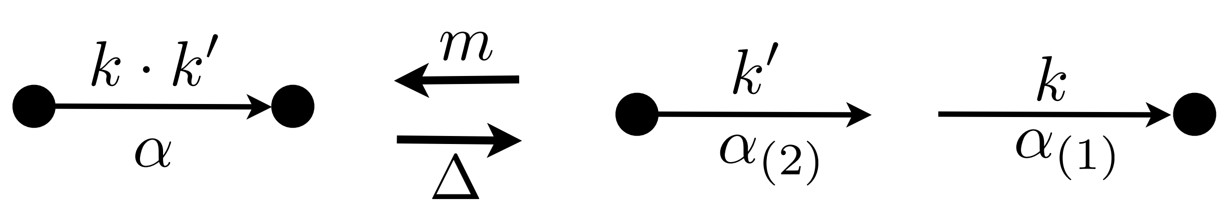
task:

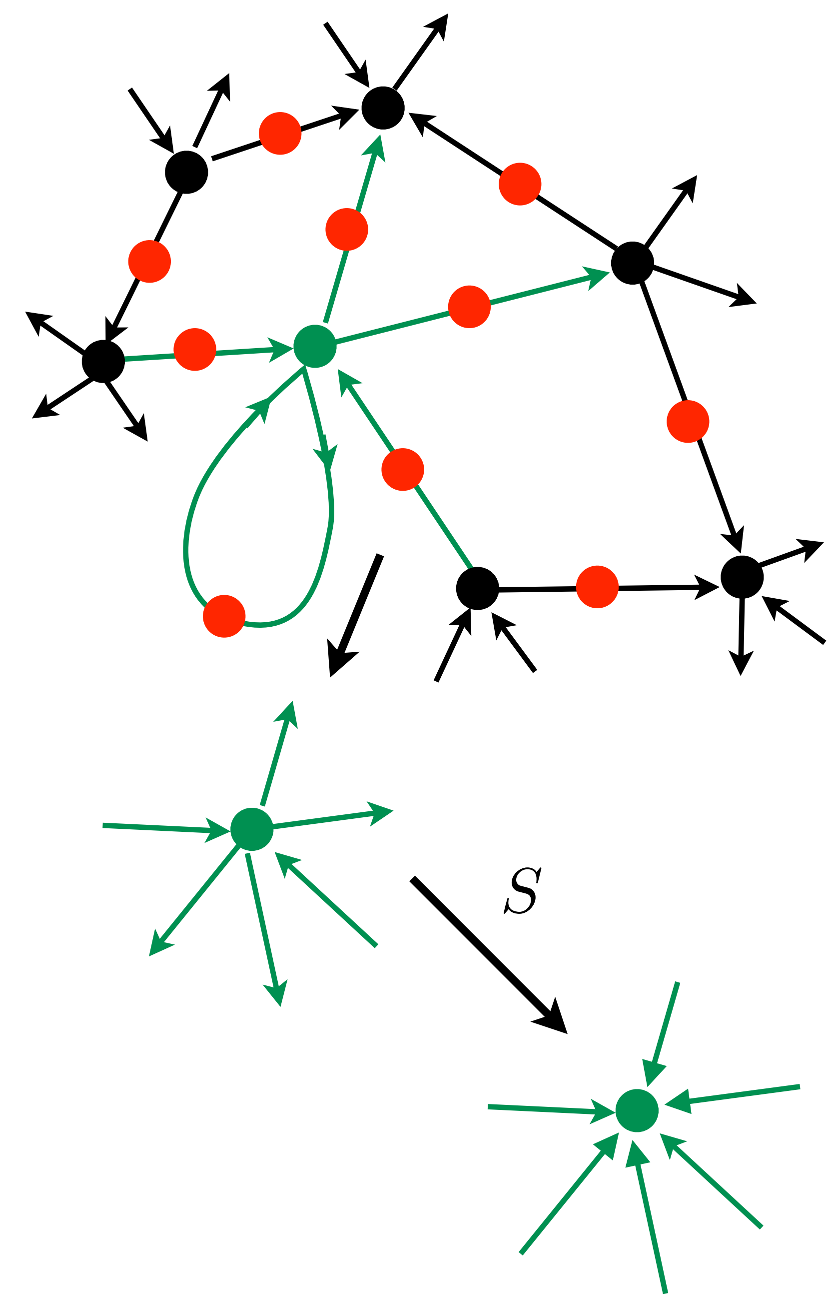
construct $K^{\otimes V}$-module algebra structure on $K^{*\otimes E}$ from K^* with left and right regular action

⇨ **impose locality condition**

$K^{\otimes V}$-module algebra structure on $K^{*\otimes E}$
induced by K -module algebra structures on vertex discs

$$G^* : K^{*\otimes E} \to K^{*\otimes 2E} \cong \otimes_{v\in V} K^{*\otimes|v|}$$





⇨ **simpler task:**

construct K-module algebra structure on $K^{*\otimes n}$
from K^* with left and right regular action

⇨ **reverse edge orientation for outgoing edges**

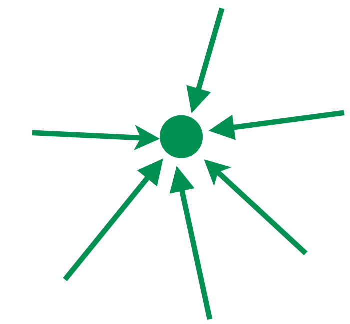
- require involution: **antipode** $S^2 = \mathrm{id}$

⇨ **simpler task:**

construct K-module algebra structure on $K^{*\otimes n}$ from K^* with left regular action

task: construct K-module algebra structure on $K^{*\otimes n}$ from K^* with left regular action

⇨ **naively:** take tensor product of algebras of edges incident at vertex



⇨ **problem:** K Hopf algebra, A_1, A_2 right module algebras over K

$\Rightarrow$ ✔ $A_1 \otimes A_2$ right module over K

✔ $A_1 \otimes A_2$ algebra

❌ **not right module algebra unless K cocommutative**

$$(a_1 \otimes a_2) \lhd k$$
$$= \big((a_1 \otimes 1)\cdot(1 \otimes a_2)\big) \lhd k = (a_1 \lhd k_{(1)} \otimes 1)\cdot(1 \otimes a_2 \lhd k_{(2)}) = a_1 \lhd k_{(1)} \otimes a_2 \lhd k_{(2)}$$
$$= \big((1 \otimes a_2)\cdot(a_1 \otimes 1)\big) \lhd k = (1 \otimes a_2 \lhd k_{(1)})\cdot(a_1 \lhd k_{(2)} \otimes 1) = a_1 \lhd k_{(2)} \otimes a_2 \lhd k_{(1)}$$

⇨ need **structure** to relate $\Delta(k) = k_{(1)} \otimes k_{(2)}$ and $\Delta^{op}(k) = k_{(2)} \otimes k_{(1)}$

⇨ require: K **quasitriangular**

Theorem [Majid] **braided tensor product of module algebras**

K quasitriangular Hopf algebra, A_1, A_2 right module algebras over K

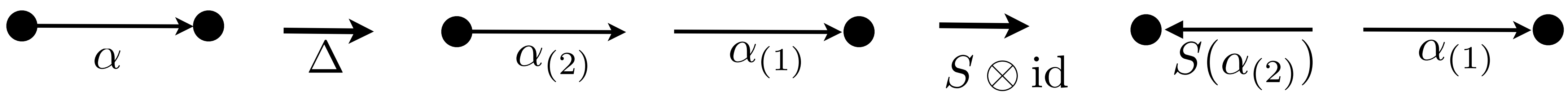
$\Rightarrow A_1 \otimes A_2$ is right module algebra over K with

- module structure $(a_1 \otimes a_2) \lhd k = (a_1 \lhd k_{(1)}) \otimes (a_2 \lhd k_{(2)})$
- algebra structure $(1 \otimes a_2)\cdot(1 \otimes a_2') = 1 \otimes a_2 a_2'$ $\quad (a_1 \otimes 1)\cdot(1 \otimes a_2) = a_1 \otimes a_2$

  $(a_1 \otimes 1)\cdot(a_1' \otimes 1) = a_1 a_1' \otimes 1$ $\quad (1 \otimes a_2)\cdot(a_1 \otimes 1) = (a_1 \otimes a_2) \lhd R$

➩ **problem:** • later combine incoming with outgoing edge ends (➩ reverted with antipode)

• antipode is anti-algebra map $S(\alpha)\cdot S(\beta) = S(\beta\cdot\alpha)$



➩ modify braided tensor product with R-matrix on the right

Hopf algebra gauge theory on a vertex disc

• **vertex disc with incoming edges**

$(\alpha)_i\cdot(\beta)_i = (\alpha\beta)_i$ $\quad i\notin I$

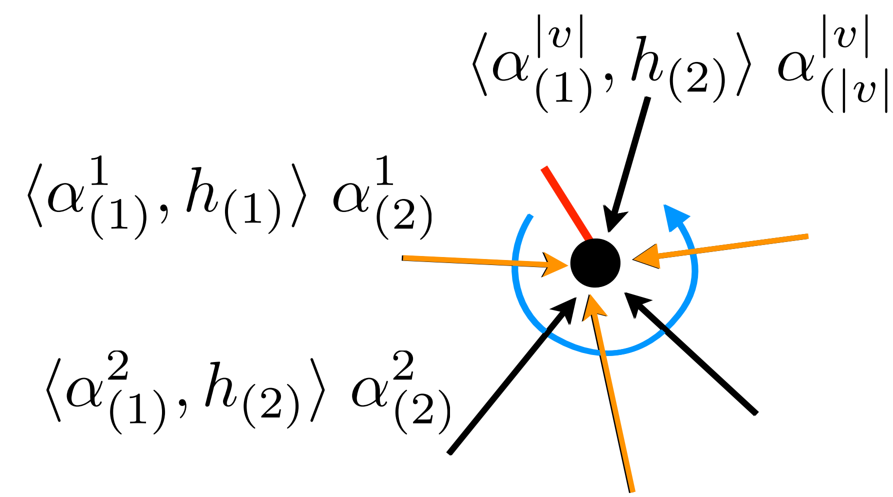
$(\alpha)_i\cdot(\beta)_i = \langle \beta_{(2)}\otimes\alpha_{(2)}, R\rangle\,(\alpha_{(1)}\beta_{(1)})_i$ $\quad i\in I$

$(\alpha)_i\cdot(\beta)_j = (\alpha\otimes\beta)_{ij}$ $\quad i<j$

$(\alpha)_i\cdot(\beta)_j = \langle \beta_{(1)}\otimes\alpha_{(1)}, R\rangle\,(\alpha_{(2)}\otimes\beta_{(2)})_{ij}$ $\quad i>j$

- algebra structure on $K^{*\otimes|v|}$
- K-module algebra with

$$(\alpha^1\otimes\ldots\otimes\alpha^n)\lhd h = \langle \alpha^1_{(1)}\cdots\alpha^n_{(1)}, h\rangle\;\alpha^1_{(2)}\otimes\ldots\otimes\alpha^n_{(2)}$$



ordering of edge ends at v

• **vertex disc with arbitrary edge orientation**

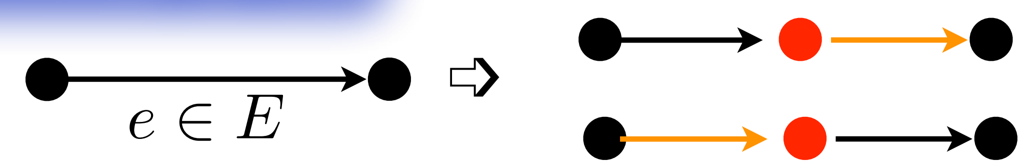
defined by condition that $S^{\tau_1}\otimes\ldots\otimes S^{\tau_n}: K^{*\otimes n}\to K^{*\otimes n}$ algebra and module isomorphism

$\tau_i = 0$ if ith edge incoming $\qquad \tau_i = 1$ if ith edge outgoing

Hopf algebra gauge theory on a ribbon graph

- **step 1**

  select one of the edge ends for each edge $e \in E$



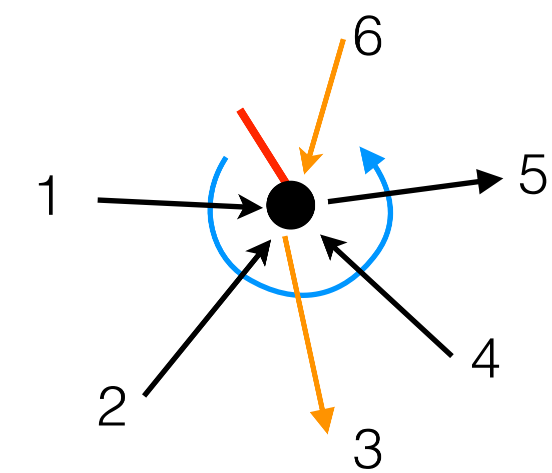
- **step 2**

  put cilium at each vertex ⇨ ordering of edge ends

  vertex $v \in V \to$ R-matrix $R_v \in K \otimes K$

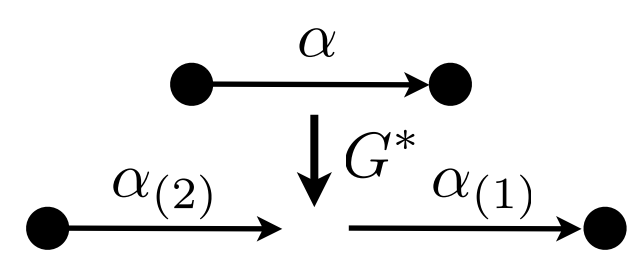
  ⇨ **Hopf algebra gauge theory on vertex discs:**

  K- right module algebra $\mathcal{A}_v$ for each vertex $v \in V$



- **step 3**

  embed $K^{*\otimes E}$ into $\otimes_{v\in V}\mathcal{A}_v$ with $G^*: K^{*\otimes E} \to K^{*\otimes 2E}$



Theorem: [Wise, C.M.]

$G^*(K^{*\otimes E}) \subset \otimes_{v\in V}\mathcal{A}_v$ is a subalgebra and a $K^{\otimes V}$-submodule

⇨ $K^{\otimes V}$- right module algebra structure on $K^{*\otimes E}$

~ algebra of functions $\mathcal{A}_\Gamma$ of Hopf algebra gauge theory

⇨ $\mathcal{A}_{\Gamma\, inv} = \{\alpha \in \mathcal{A}_\Gamma : \alpha \lhd h = \epsilon(h)\,\alpha\ \forall h \in K^{\otimes V}\} \subset \mathcal{A}_\Gamma$ is subalgebra

~ algebra of observables of Hopf algebra gauge theory

- independent of choice of cilia
- if $R_v = R$ for all $v \in V$: independent of choice of edge ends

Hopf algebra gauge theory on a ribbon graph

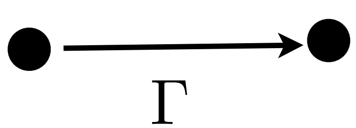
- **Ex: group algebras of finite groups**

  $K = \mathbb{F}[G]$, any ribbon graph Γ ⇨ group lattice gauge theory for G on Γ

- **Ex: cocommutative Hopf algebras**

  K cocommutative, any ribbon graph Γ ⇨ $\mathcal{A}_\Gamma = K^{*\otimes E}$ commutative

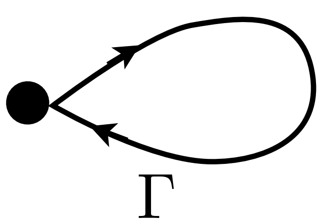
- **Ex:**



$$\alpha \cdot \beta = \langle \beta_{(2)} \otimes \alpha_{(2)}, R\rangle\, \alpha_{(1)}\beta_{(1)}$$

if $K = D(H)$ ⇨ Heisenberg double $\mathcal{A}_\Gamma \cong H\#H^*$

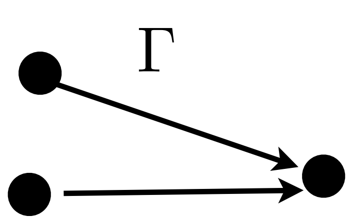
- **Ex:**



$$\alpha \cdot \beta = \langle \beta_{(1)} \otimes S(\alpha_{(3)}), R\rangle \langle \beta_{(2)} \otimes \alpha_{(1)}, R\rangle\, \beta_{(3)}\alpha_{(2)}$$

if $K = D(H)$ ⇨ Drinfeld double $\mathcal{A}_\Gamma \cong D(H) = K$

- **Ex:**



$$(\alpha \otimes \gamma) \cdot (\beta \otimes \delta) = \langle \beta_{(1)} \otimes \gamma_{(1)}, R\rangle \langle \beta_{(3)} \otimes \alpha_{(2)}, R\rangle \langle \delta_{(2)} \otimes \gamma_{(3)}, R\rangle\, \alpha_{(1)}\beta_{(2)} \otimes \gamma_{(2)}\delta_{(1)}$$

- **general result**

$\mathcal{A}_\Gamma \cong$ lattice algebra from combinatorial quantisation of Chern-Simons theory

[Alekseev, Grosse, Schomerus '94] , [Buffenoir Roche '95]

4. Curvature and flatness

➪ **flatness condition at faces of the graph: require notion of holonomy**

- **holonomy:** path $p \in \mathcal{P}(\Gamma)$ $\rightarrow$ linear map $\mathrm{Hol}_p : K^{\otimes E} \rightarrow K$
  $\rightarrow$ linear map $\mathrm{Hol}_p^* : K^* \rightarrow K^{*\otimes E}$

in duality
$\langle \mathrm{Hol}_p(k), \alpha \rangle = \langle k, \mathrm{Hol}_p^*(\alpha) \rangle$

composition of paths $p \circ q$ $\rightarrow$ $\mathrm{Hol}_p^* \bullet \mathrm{Hol}_q^*$ product of holonomies

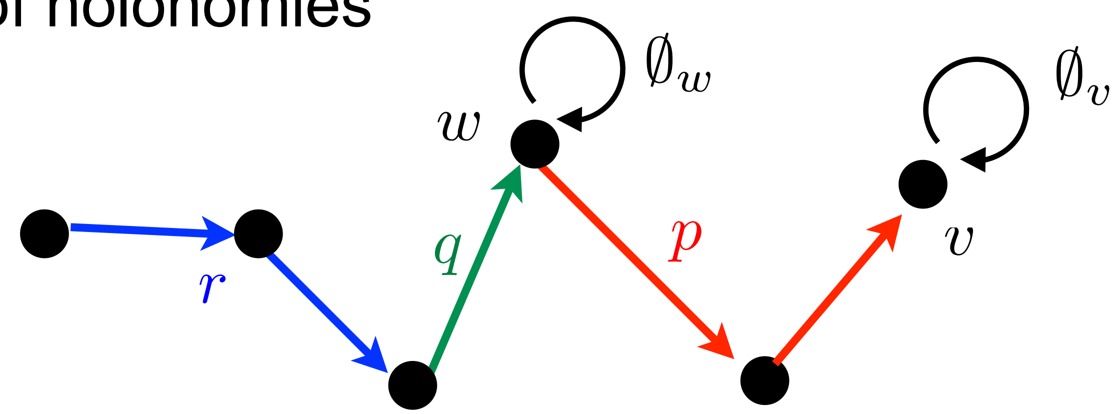
consistency conditions

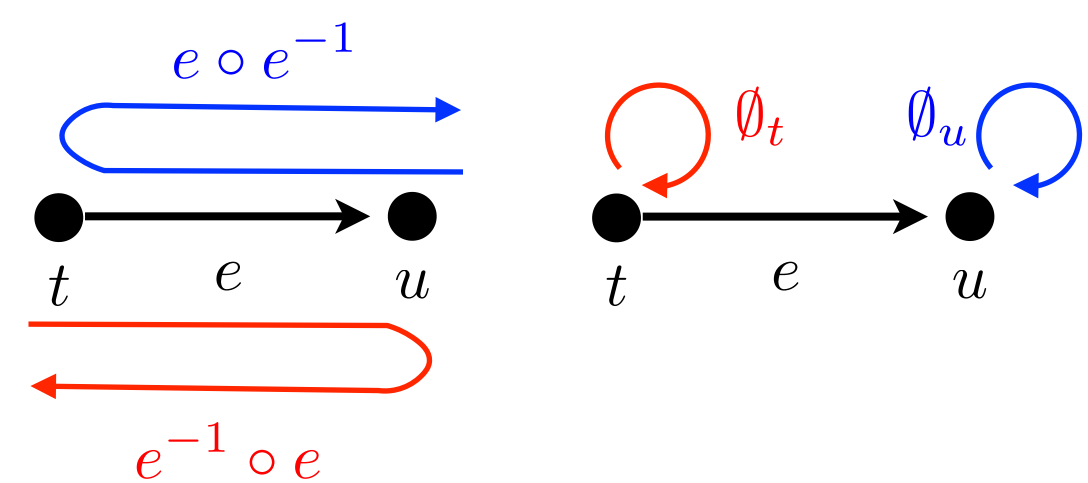
$p \circ (q \circ r) = (p \circ q) \circ r$ ➪ • associative

$p \circ \emptyset_v = \emptyset_w \circ p = p$ ➪ unit for •

$e \circ e^{-1} = \emptyset_u \quad e^{-1} \circ e = \emptyset_t$ ➪ Hol_e^* has an inverse

➪ need: algebra structure on $\mathrm{Hom}_{\mathbb{F}}(K^*, K^{*\otimes E})$
such that Hol_e^* has inverse





Def

Let (C, Δ, ϵ) be a coalgebra and $(A, m, 1)$ an algebra. The **convolution product** on $\mathrm{Hom}_{\mathbb{F}}(C, A)$ is given by $\phi \bullet \psi = m \circ (\phi \otimes \psi) \circ \Delta$. It is associative and unital with unit $\epsilon\, 1$.

- **here:** $C = K^* \quad A = K^{*\otimes E}$
  holonomies of reversed edges from antipode of K^*

- **fact:**

The consistency conditions are satisfied for

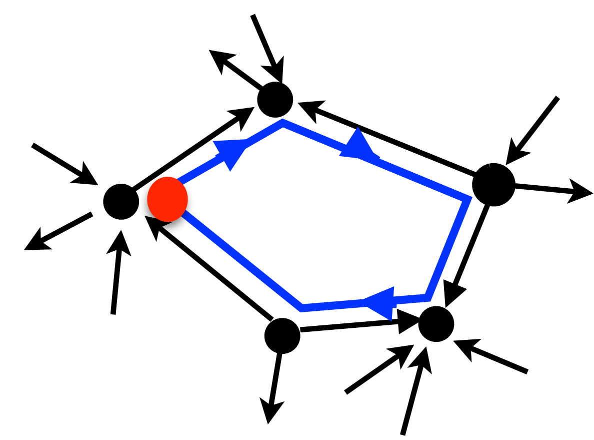
$\mathrm{Hol}^*_{\emptyset_v}(\alpha) = \epsilon(\alpha)\, 1^{\otimes E} \quad \forall v \in V$ $\qquad \mathrm{Hol}^*_e(\alpha) = (\alpha)_e \quad \forall e \in E$

$\mathrm{Hol}^*_{p\circ q}(\alpha) = \mathrm{Hol}^*_p(\alpha_{(1)}) \cdot \mathrm{Hol}^*_q(\alpha_{(2)}) \quad \forall p, q \in \mathcal{P}(\Gamma)$ $\qquad \mathrm{Hol}^*_{e^{-1}}(\alpha) = (S(\alpha))_e$

This defines a holonomy functor $\mathrm{Hol}^* : \mathcal{P}(\Gamma) \to \mathrm{Hom}_{\mathbb{F}}(K^*, K^{*\otimes E})$

Def

Let $\mathcal{A}_\Gamma$ be the algebra of functions of a Hopf algebra gauge theory on Γ .
The **curvature of a face** $f \in F$ is the holonomy $\mathrm{Hol}^*_f : K^* \to K^{*\otimes E}$



Theorem: [Alekseev, Grosse, Schomerus], [Buffenoir, Roche], [Wise, C.M.]

- The holonomy of each face $f \in F$ defines an algebra homomorphism $\mathrm{Hol}^*_f : \{\alpha \in K^* : \Delta(\alpha) = \Delta^{op}(\alpha)\} \to Z(\mathcal{A}_{\Gamma\, inv})$
- The Haar integral $\eta \in K^*$ defines commuting projectors $P_f : \mathcal{A}_{\Gamma\, inv} \to \mathcal{A}_{\Gamma\, inv}$, $\alpha \mapsto \mathrm{Hol}^*_f(\eta) \cdot \alpha$
- Their image $\mathcal{M} = P_1 \circ ... \circ P_F(\mathcal{A}_{\Gamma\, inv})$ is a topological invariant.
  It depends only on the surface S obtained by gluing discs to the faces of Γ.

⇨ **Hopf algebra analogue of moduli space of flat G -connections**

$$\mathrm{Hom}(\pi_1(S), G)/G = \{g \in G^{\times E} : c^f(g) = 1 \forall f \in F\}/G^{\times V}$$