Lecture series on 3d gravity

Lecture 2:
Phase space and Symplectic Structure

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3d gravity - symplectic structure

symplectic structure on phase space of 3d gravity $\mathcal{M}_\Lambda(S)$

from moduli space of flat G-connections on S

- for compact surface S of genus $g > 1$: related to moduli space of flat G_Λ- connections on S
- **moduli space of flat G-connections** $\mathrm{Hom}(\pi_1(S), G)/G$
  - canonical symplectic structure [Goldman, Atiyah, Bott]
  - description in terms of Poisson-Lie groups [Fock, Rosly, Alekseev, Malkin]

    ➪ classical counterpart of quantum group symmetries

    ➪ starting point for combinatorial quantisation [Alekseev, Grosse, Schomerus, Buffenoir, Roche]
- phase spaces $\mathcal{M}_\Lambda(S)$ of 3d gravity: $\mathcal{M}_\Lambda(S) \subset \mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$

from Teichmüller space

- for cusped surface S: related to Teichmüller space

$$\mathcal{T}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}_0(S) = \mathrm{Hom}_F(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

- parametrisation by shear coordinates associated with ideal triangulation [Fock-Checkov, Penner]
- simple description of symplectic structure via shear coordinates: Weil-Petersson form
- phase spaces $\mathcal{M}_\Lambda(S)$ of 3d gravity: via analytic continuation of shear coordinates

1. Symplectic structure on moduli space of flat G-connections

Goldman's symplectic structure [Goldman]

- S compact surface of genus $g > 0$
- Lie group G
- non-degenerate, symmetric, Ad-invariant bilinear form κ on $\mathfrak{g} = \mathrm{Lie}\, G$

$$\kappa_{ab} = \kappa(T_a, T_b) \qquad \kappa^{ab}\kappa_{bc} = \delta^a_c$$

- **symplectic structure on $\mathrm{Hom}(\pi_1(S), G)/G$**

- **Wilson loop observables**

$\left.\begin{array}{l} \lambda \in \pi_1(S) \\ f \in C^\infty(G)^G \end{array}\right\} \Rightarrow$ function $f_\lambda : \mathrm{Hom}(\pi_1(S), G)/G \to \mathbb{R}$

$$\rho \in \mathrm{Hom}(\pi_1(S), G) \mapsto f_\lambda(\rho) = f(\rho(\lambda))$$

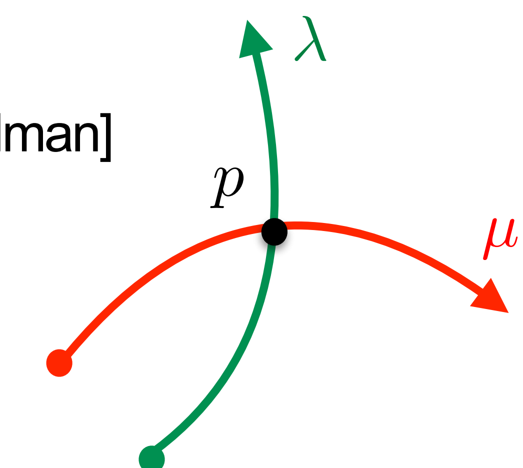
- **Poisson bracket**

$\lambda, \mu \in \pi_1(S)$

$f, h \in C^\infty(G)^G$

$$\{f_\lambda, h_\mu\} = \sum_{p \in \lambda \cap \mu} \epsilon(\lambda, \mu, p)\kappa^{ab}\, \nabla^L_a f_{\lambda_p}\, \nabla^L_b h_{\mu_p}$$

[Goldman]



oriented intersection #

right invariant vector fields for basis $\{T_a\}$ of $\mathfrak{g}$

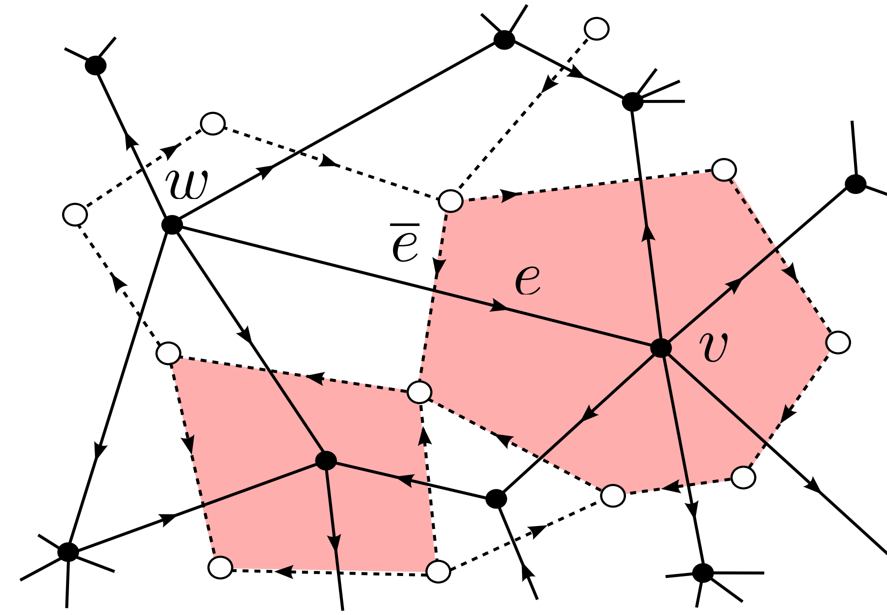
$$\nabla^L_a f(g) = \frac{d}{dt}\Big|_{t=0} f(e^{-tT_a} g)$$

disadvantages:

- difficult to use in concrete computations
- difficult to quantise

graph gauge theoretical description

[Fock, Rosly, Alekseev, Malkin]



S compact surface

- non-degenerate directed graph Γ embedded in S
- cut S along dual graph $\overline{\Gamma}$ ⇨ for each vertex $v \in V$: polygon P_v

- **description of flat G -connections on S**

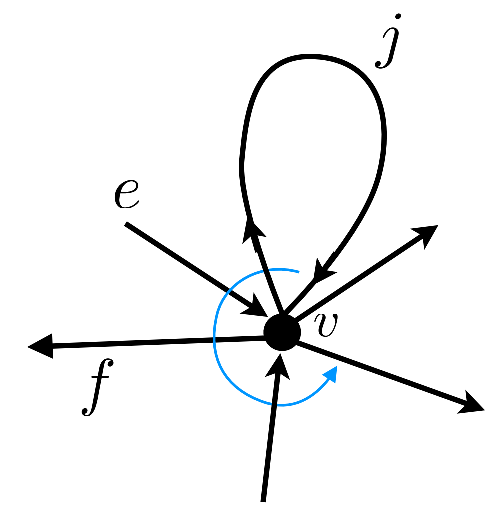
- **trivialise** connection on each polygon P_v

  $A|_{P_v} = \gamma_v^{-1} d\gamma_v$

- **transition functions** $e \in E \to g_e \in G$

  $A|_{\bar{e}} = \gamma_v^{-1} d\gamma_v|_{\bar{e}} = \gamma_w^{-1} d\gamma_w|_{\bar{e}} \quad \Rightarrow \quad \gamma_v|_{\bar{e}} = g_e \cdot \gamma_w|_{\bar{e}}$ with $g_e \in G$

- **flatness conditions** $f \in F \to c^f = \Pi_{e \in f}\, g_e^{\epsilon_e} \overset{!}{=} 1$



- **"gauge" transformations** $v \in V \to g_v \in G$

  $\gamma_v \to g_v \cdot \gamma_v$ with $g_v \in G \quad \Rightarrow A|_{P_v} \to A|_{P_v}$

- **action on transition functions** $g_e \to g_v g_e g_w^{-1}$

$g_e \mapsto g_v \cdot g_e$
$g_f \mapsto g_f \cdot g_v^{-1}$
$g_j \mapsto g_v \cdot g_j \cdot g_v^{-1}$

- **moduli space of flat G - connections**

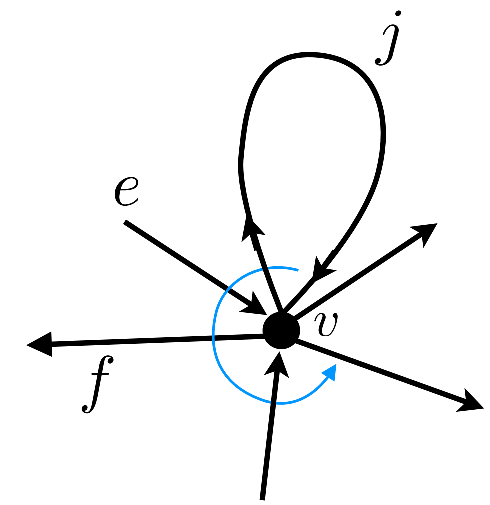
$$\mathrm{Hom}(\pi_1(S), G)/G = \{\text{flat } G\text{-connections on } S\}/\{\text{gauge transformations}\}$$
$$= \{g \in G^{\times E} : c^1(g) = \ldots = c^F(g) = 1\}/G^{\times V}$$

graph gauge theory

setting: G Lie group

Γ directed ribbon graph = directed graph with cyclic ordering of edge ends at vertex

- **graph gauge fields** — elements of $G^{\times E}$
- **graph gauge transformations** — elements of $G^{\times V}$
- **action on gauge fields** — group action $\triangleright : G^{\times V} \times G^{\times E} \to G^{\times E}$
- **functions of gauge fields** — elements of $C^\infty(G^{\times E})$
- **observables** — elements of $C^\infty(G^{\times E})^{G^{\times V}}$
- **flatness conditions** — $c^f = \Pi_{e\in F}\, g_e^{\epsilon_e} \overset{!}{=} 1$ for each face $f \in F$
- **moment map** — $\mu = (c^1, ..., c^F) : G^{\times E} \to G^{\times F}$
- **moduli space** — $\mathrm{Hom}(\pi_1(S), G)/G = \mu^{-1}(1)/G^{\times V}$



$g_e \mapsto g_v \cdot g_e$

$g_f \mapsto g_f \cdot g_v^{-1}$

$g_j \mapsto g_v \cdot g_j \cdot g_v^{-1}$

task: Poisson structure on $G^{\times E}$ that induces Poisson structure on $\mathrm{Hom}(\pi_1(S), G)/G$

⇨ require:
- observables form Poisson subalgebra
- class functions of flatness conditions Poisson-commute with all observables

  $\{h \circ c^f, k\} \overset{!}{=} 0$ for all faces $f \in F$, observables k and $h \in C^\infty(G)^G$

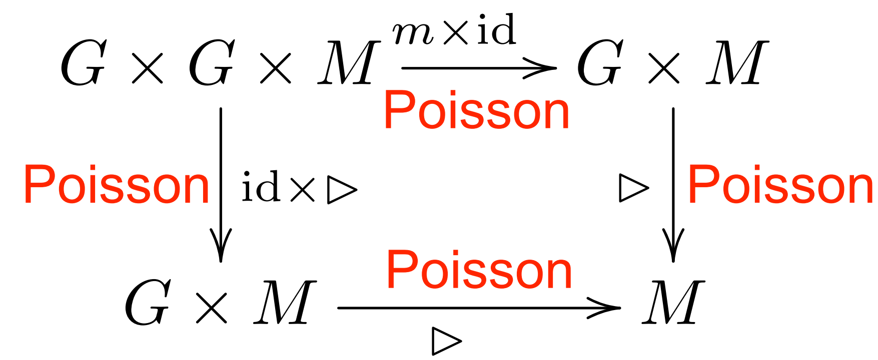
⇨ then: $\mathrm{Hom}(\pi_1(S), G)/G = \mu^{-1}(1)/G^{\times V}$ inherits Poisson structure via Poisson reduction

graph gauge theory - abstract picture

- **gauge fields:** Poisson manifold $(M, \{\,,\,\}_M)$
- **gauge transformations:** Lie group G
- **action on gauge fields:** group action $\rhd : G \times M \to M$
- **observables:** $C^\infty(M)^G$

Poisson subalgebra of observables ?

⇨ need:
- Poisson structure on G
- group action $\rhd : G \times M \to M$ is Poisson map
- multiplication map $m : G \times G \to G$ is Poisson map



Def

- A **Poisson-Lie group** is a Lie group G with a Poisson structure $\{\,,\,\}_G$ such that $m : G \times G \to G$ is a Poisson map
- A **Poisson G-space** is a Poisson-manifold $(M, \{\,,\,\}_M)$ with a G-action $\rhd : G \times M \to M$ that is a Poisson map.

similarly: Poisson G -spaces for right actions

- **Ex:** $G_1, ..., G_n$ **Poisson-Lie groups** $\Rightarrow G_1 \times ... \times G_n$ Poisson-Lie group
- **Ex: Poisson-Lie group** G $\Rightarrow$ Poisson $G \times G$ - space for left and right multiplication
- **Ex: Poisson homogeneous space**
  G Poisson-Lie group, $H \subset G$ Poisson-Lie subgroup $\Rightarrow$ G/H Poisson G-space

- **fact:** M Poisson G-space $\Rightarrow$ $C^\infty(M)^G$ Poisson subalgebra of $C^\infty(M)$

$$\underbrace{\{f_1,f_2\}_M(g\rhd m)}_{=\{f_1,f_2\}_M(m)} = \underbrace{\{f_1(g\rhd -), f_2(g\rhd -)\}_M(m)}_{=\{f_1,f_2\}_M(m)} + \underbrace{\{f_1(-\rhd m), f_2(-\rhd m)\}_G(g)}_{=0} \qquad \forall f_1,f_2\in C^\infty(M)^G$$

Poisson structure for graph gauge theory

task:

construct Poisson $G^{\times V}$- space structure on $G^{\times E}$ from Poisson $G\times G$-space structure on G

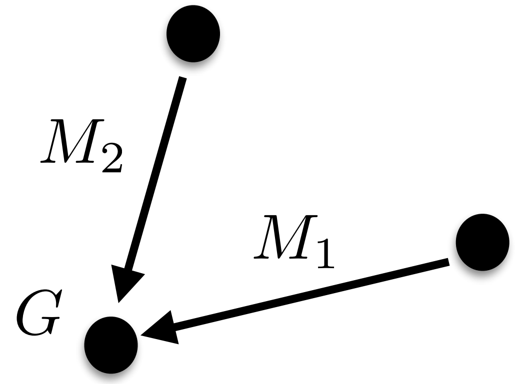
- **gauge transformations** Poisson-Lie group $G^{\times V}$
- **gauge fields** naively: product of Poisson G-space structures on G

problem: G Poisson-Lie group, M_1, M_2 Poisson G - spaces

$\Rightarrow$ ✔ $M_1\times M_2$ is Poisson manifold

✔ group action $\rhd: G\times(M_1\times M_2)\to M_1\times M_2$

❌ **not a Poisson G- space**



for $f_i = F_i\circ\pi_i$ with $\pi_i: M_1\times M_2\to M_i$ and $F_i\in C^\infty(M_i)$:

$$\underbrace{\{f_1,f_2\}(g\rhd_1 m_1, g\rhd_2 m_2)}_{=0} \neq \underbrace{\{f_1(g\rhd_1 -), f_2(g\rhd_2 -)\}(m_1,m_2)}_{=0} + \underbrace{\{f_1(-\rhd_1 m_1), f_2(-\rhd_2 m_2)\}_G(g)}_{\neq 0}$$

⇨ **modify Poisson bracket, need additional structure on G**

quasitriangular Poisson-Lie groups

Def A **Lie bialgebra** is a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ with an antisymmetric linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, the **cocommutator**, satisfying

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta \qquad \text{(coassociativity)}$$

$$\delta([\mathbf{x}, \mathbf{y}]) = (\mathrm{ad}_{\mathbf{x}} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{\mathbf{x}})\delta(\mathbf{y}) - (\mathrm{ad}_{\mathbf{y}} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{\mathbf{y}})\delta(\mathbf{x}) \qquad \text{(cocycle condition)}$$

- **fact:** G Poisson-Lie group $\Rightarrow$ $\mathfrak{g} = T_e G$ Lie bialgebra

multiplication $\cdot_G$ $\Rightarrow$ Lie bracket $[\ ,\ ]_{\mathfrak{g}}$

Poisson bracket $\{\ ,\ \}_G$ $\Rightarrow$ cocommutator $\delta_{\mathfrak{g}}$ $\quad \langle \delta(\mathbf{x}), d_e f \otimes d_e g \rangle = \langle \mathbf{x}, d_e\{f, g\} \rangle$

Def A Poisson-Lie group $(G, \{\ ,\ \})$ is **quasitriangular** if it has a **classical r-matrix** $r \in \mathfrak{g} \otimes \mathfrak{g}$ with $\delta(\mathbf{x}) = (\mathrm{ad}_{\mathbf{x}} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{\mathbf{x}})(r)$ (coboundary)

symmetric part $r_{(s)} = r + \sigma(r)$ Ad-invariant

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \qquad \text{(CYBE)}$$

- **Ex: semidirect products** $G \ltimes \mathfrak{g}^*$
  - group multiplication $(g, \mathbf{x}) \cdot (h, \mathbf{y}) = (gh, \mathbf{x} + \mathrm{Ad}^*(g)\mathbf{y})$
  - quasitriangular with $r = T_a \otimes t^a \in (\mathfrak{g} \oplus \mathfrak{g}^*) \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)$
  - Poisson-Lie structure ~ Lie bracket of $G \times \mathfrak{g}^*$

$\{T_a\}$ basis of $\mathfrak{g}$

$\{t^a\}$ dual basis of $\mathfrak{g}^*$

$$\{T_a, T_b\} = [T_a, T_b] = f_{ab}{}^c T_c \qquad \{f, T_a\} = 0 \qquad \{f, g\} = 0 \quad f, g \in C^\infty(G)$$

Poisson spaces for quasitriangular Poisson-Lie groups

- **fact:**

G quasitriangular Poisson-Lie group, M_1, M_2 Poisson G -spaces
$\Rightarrow M_1 \times M_2$ with deformed bracket is Poisson G -space

$$\{f_1, f_2\}_r = \{f_1, f_2\}_{M_1\times M_2} + r^{ab}(\nabla_a^{M_1} f_1\, \nabla_b^{M_2} f_2 - \nabla_a^{M_1} f_2\, \nabla_b^{M_2} f_1)$$

← vector fields for group actions

↑ $r = r^{ab} T_a \otimes T_b \in \mathfrak{g}\otimes\mathfrak{g}$

$$\nabla_a^{M_i} f(m) = \frac{d}{dt}|_{t=0} f(e^{-tT_a} \rhd m)$$

- **fact:**

G quasitriangular Poisson-Lie group

$\Rightarrow$ Poisson-Lie structure on G is $\{f,g\} = r^{ab}_{(a)}(\nabla^L_a f \nabla^L_b g - \nabla^R_a f \nabla^R_b g)$ (Sklyanin bracket)

$\Rightarrow G$ is Poisson $G\times G$ - space with $\{f,g\} = r^{ab}_{(a)}(\nabla^L_a f \nabla^L_b g + \nabla^R_a f \nabla^R g)$ (Heisenberg double)

$\Rightarrow G$ is Poisson G - space with respect to conjugation action with

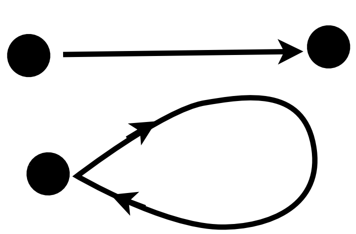
$$\{f,g\} = r^{ab}_{(a)}(\nabla^L_a f \nabla^L_b g + \nabla^R_a f \nabla^R g) + r^{ab}(\nabla^L_a f \nabla^R_b g - \nabla^L_a g \nabla^R_b f)$$ (dual bracket)

↑↑ right and left invariant vector fields

$$\nabla^L_a f(g) = \frac{d}{dt}|_{t=0} f(e^{-tT_a} g) \qquad \nabla^R_a f(g) = \frac{d}{dt}|_{t=0} f(g e^{tT_a})$$

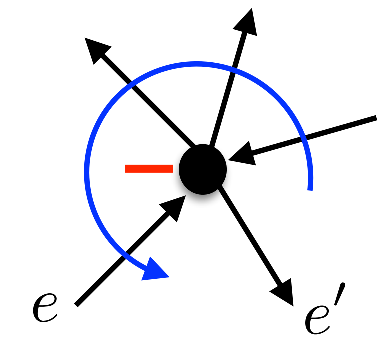
⇨ **structures for graph gauge theory:**

- vertex $v\in V$ → quasitriangular Poisson-Lie group G
- open edge $e \in E$ → Heisenberg double
- loop $e\in E$ → dual bracket
- combine structures for edges at vertex $v \in V$ with deformed bracket



Poisson structure for a graph gauge theory [Fock,Rosly]

ingredients:
- directed graph with linear ordering of edge ends at vertices
- quasitriangular Poisson-Lie group (G, r)



⇨ Poisson structure in terms of functions that depend only on edge $e \in E$

$f_e = f \circ \pi_e \in C^\infty(G^{\times E})$ with $f \in C^\infty(G)$ and $\pi_e : G^{\times E} \to G$, $(g_1, .., g_E) \mapsto g_e$

(*)

$$\{f_e, h_e\} = r^{ab}_{(a)}(\nabla^L_a f_e \nabla^L_b h_e + \nabla^R_a f_e \nabla^R h_e)$$

$$\{f_e, h_{e'}\} = r^{ab} \nabla^X_a f_e \nabla^X_b h_{e'} \qquad e, e' \in E(v), e < e'$$

$$\{f_e, h_{e'}\} = 0 \qquad \{s(e), t(e)\} \cap \{s(e'), t(e')\} = \emptyset$$

$X = L$ incoming

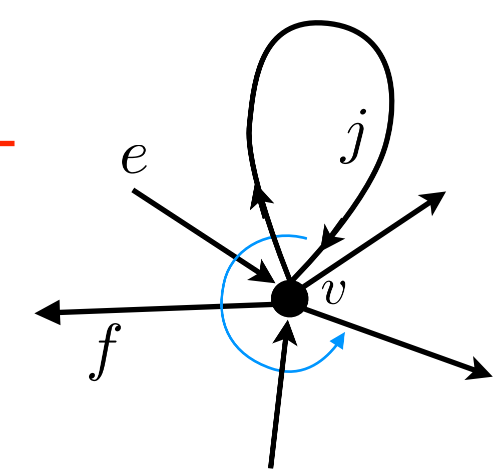
$X = R$ outgoing

$$\nabla^L_a f(g) = \frac{d}{dt}|_{t=0} f(e^{-tT_a} g)$$

$$\nabla^R_a f(g) = \frac{d}{dt}|_{t=0} f(g e^{tT_a})$$

Theorem: [Fock,Rosly]

- (*) defines a Poisson $G^{\times V}$-space structure on $G^{\times E}$
- observables form Poisson subalgebra $C^\infty(G^{\times E})^{G^{\times V}} \subset C^\infty(G^{\times E})$
- Poisson bracket of observables depends only on $r_{(s)} = r + \sigma(r)$
- for all faces $f \in F$ and $h \in C^\infty(G)^G$: $\{h \circ c^f, k\} = 0$ for all observables k
- induces symplectic structure on $\mathrm{Hom}(\pi_1(S), G)/G = \mu^{-1}(1)/G^{\times V}$
- coincides with Goldman's symplectic structure if $r^{ab}_{(s)} = \kappa^{ab}$



$$g_e \mapsto g_v \cdot g_e$$

$$g_f \mapsto g_f \cdot g_v^{-1}$$

$$g_j \mapsto g_v \cdot g_j \cdot g_v^{-1}$$

application to 3d gravity

- **description of isometry groups in terms of** $R_\Lambda = (\mathbb{R}^2, +, \cdot_\Lambda)$
  $(a,b)\cdot_\Lambda (c,d) = (ac - \Lambda bd, ad + bc)$

- **isometry groups** $G_\Lambda = \{M \in \mathrm{Mat}(2, R_\Lambda) : \det(M) = 1\}$

- **Lie algebras** $\mathfrak{g}_\Lambda = \{M \in \mathrm{Mat}(2, R_\Lambda) : \mathrm{tr}(M) = 0\}$

basis of $\mathfrak{g}_\Lambda$ $J_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $J_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $J_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Lorentz transformations $\mathfrak{sl}(2,\mathbb{R}) = \langle J_0, J_1, J_2\rangle_\mathbb{R}$

$P_0 = \ell\, J_0$ $P_1 = \ell\, J_1$ $P_2 = \ell\, J_2$ translations

Lie bracket $[J_a, J_b] = \epsilon_{ab}{}^c J_c$ $[J_a, P_b] = \epsilon_{ab}{}^c P_c$ $[P_a, P_b] = -\Lambda \epsilon_{ab}{}^c J_c$

Killing form $K(X,Y) = \mathrm{tr}(X \cdot Y)$

⇨ **for 3d gravity:** Goldman's symplectic structure for $\kappa = \mathrm{Im}_\ell\, K$ ⇨ $\kappa(J_a, J_b) = \kappa(P_a, P_b) = 0$
$\kappa(J_a, P_b) = \eta_{ab}$

- **r-matrix for** $\mathfrak{sl}(2,\mathbb{R})$ $r = J_a \otimes J^a + n^a \epsilon_{abc}\, J^b \otimes J^c$ with $\mathbf{n} \in \mathbb{R}^3,\ \mathbf{n}^2 = 1$

- **r-matrices for** $\mathfrak{g}_\Lambda$ $r = P_a \otimes J^a + n^a \epsilon_{abc} J^b \otimes J^c$ with $\mathbf{n} \in \mathbb{R}^3,\ \mathbf{n}^2 = -\Lambda$

⇨ **quasitriangular Poisson-Lie group structures for 3d gravity**

- **Ex: Λ=0** $G_\Lambda = \mathrm{Iso}(2,1) \cong \mathrm{PSL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$ $\mathfrak{g}_\Lambda = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3$

  choose $\mathbf{n} = 0$ ⇨ $r = P_a \otimes J^a$

⇨ in terms of functions $f, g \in C^\infty(\mathrm{PSL}(2,\mathbb{R}))$ and basis $\{J_a\}$ of $\mathfrak{sl}(2,\mathbb{R})$

Sklyanin bracket	$\{J_a, J_b\} = \epsilon_{ab}{}^c J_c$	$\{f,g\} = 0$	$\{J_a, f\} = 0$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3$
dual bracket	$\{J_a, J_b\} = \epsilon_{ab}{}^c J_c$	$\{f,g\} = 0$	$\{J_a, f\} = (\nabla^L_a + \nabla^R_a) f$	$\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3$
Heisenberg double	$\{J_a, J_b\} = \epsilon_{ab}{}^c J_c$	$\{f,g\} = 0$	$\{J_a, f\} = \nabla^L_a f$	$T^*\mathrm{PSL}(2,\mathbb{R})$

- **interpretation in 3d gravity**

in all brackets: $f, g \in C^\infty(\mathrm{PSL}(2,\mathbb{R}))$ ~ functions of momentum coordinates

⇨ Poisson commute for $\Lambda = 0$

⇨ addition of momenta via group multiplication

Heisenberg double $J_0, J_1, J_2 \in \mathrm{sl}(2,\mathbb{R})$ ~ position coordinates

⇨ non-commutative position coordinates

⇨ Poisson bracket = Lie bracket for $\mathfrak{sl}(2,\mathbb{R})$

⇨ bracket between momenta and positions is deformed

dual bracket $J_0, J_1, J_2 \in \mathrm{sl}(2,\mathbb{R})$ ~ angular momenta

⇨ dual bracket describes isometry algebra of M_3

Poisson structure and spacetime geometry - gauge theoretical picture

- **questions:**
  geometrical interpretation of Poisson structures on $G^{\times E}$ and $\mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$?
  geometrical interpretation of Wilson loop observables?

- **Wilson loop observables**
  $\lambda \in \pi_1(S)$ ⇨ two fundamental observables on $\mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$: $m_\lambda : \rho \mapsto \mathrm{Re}_\ell\, \mathrm{tr}(\rho(\lambda))$
  $s_\lambda : \rho \mapsto \mathrm{Im}_\ell\, \mathrm{tr}(\rho(\lambda))$

- **relation to geodesics**
  conformally static spacetimes characterised by group homomorphism $\rho_0 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$
  $\lambda \in \pi_1(S)$ simple ⇨ corresponds to unique geodesic λ on $\Sigma = \mathbb{H}^2/\mathrm{im}(\rho_0)$

- **earthquake and grafting**
  transformation of holonomies under earthquake and grafting along geodesic λ induces flows

$$\mathrm{Qu}_{w\lambda}, \mathrm{Gr}_{w\lambda} : \mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda \to \mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$$

Theorem: [C.M.]

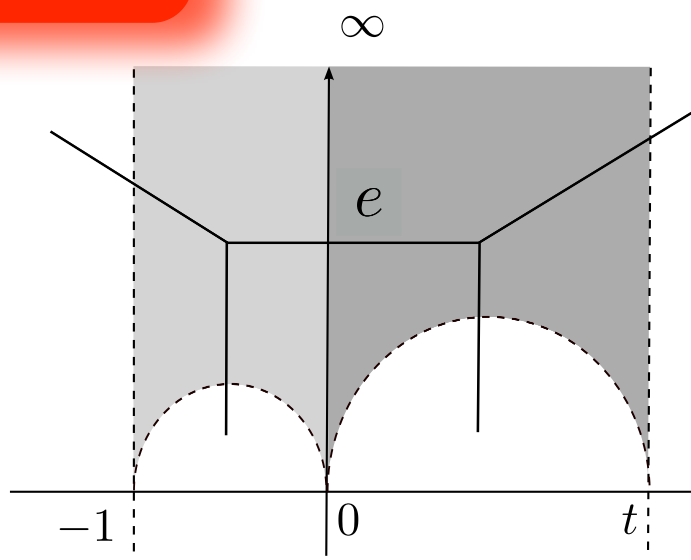
If $\lambda \in \pi_1(S)$ is simple, then the observables s_λ, m_λ are the Hamiltonians for the earthquake and grafting flows on $\mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$

$$\{s_\lambda, g\} = \frac{d}{dt}\Big|_{t=0} g \circ \mathrm{Qu}_{t\lambda} \qquad \{m_\lambda, g\} = \frac{d}{dt}\Big|_{t=0} g \circ \mathrm{Gr}_{t\lambda}$$

⇨ **fundamental observables for $\lambda \in \pi_1(S)$ generate the two fundamental geometrical transformations that change the geometry of the spacetime**

2. Symplectic structure from Teichmüller space

S oriented surface of genus g with s>0 cusps, 2g-2+s>0



- **shear coordinates**

  trivalent graph Γ dual to ideal triangulation

  lifts to ideal triangulation of $\mathbb{H}^2$

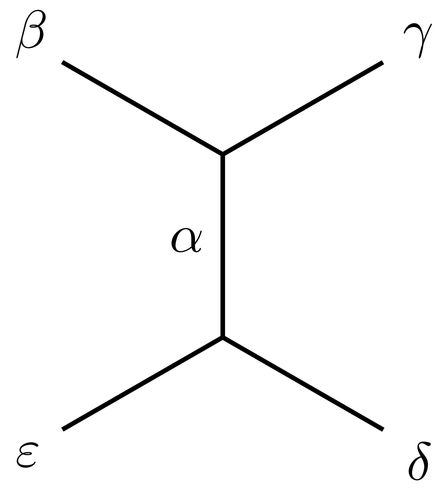
  edge $e \in E$ ➩ cross ratio of ideal square $x_e(h) = \log t$

  face $f \in F$ ➩ constraint $c^f = \sum_{\alpha \in f} \theta^f_\alpha x^\alpha$ $\theta^f_\alpha \in \{1, 2\}$ - multiplicity of α in f

- **Teichmüller space**

  moment map $c = (c^1, ..., c^F) : \mathbb{R}^E \to \mathbb{R}^F$

  Teichmüller space $\mathcal{T}(S) \cong \ker(c) \subset \mathbb{R}^E$



- **symplectic structure in shear coordinates**

  $$\{x^\alpha, x^{\alpha'}\} = \pi^{\alpha\alpha'}_{WP} = \delta^{\beta\alpha'} + \delta^{\delta\alpha'} - \delta^{\gamma\alpha'} - \delta^{\epsilon\alpha'}$$

  constraints Poisson-commute with shear coordinates $\{c^f, x^\alpha\} = 0$

Theorem: [Fock-Checkov, Penner]

The symplectic quotient of $(\mathbb{R}^E, \{\,,\,\})$ with respect to $c = (c^1, ..., c^F) : \mathbb{R}^E \to \mathbb{R}^F$ is symplectomorphic to Teichmüller space with the Weil-Petersson symplectic structure and to Goldman's symplectic structure on $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$

symplectic structure of 3d gravity in generalised shear coordinates

S oriented surface of genus g with s>0 cusps, 2g-2+s>0

- **generalised shear coordinates**

  trivalent graph Γ dual to ideal triangulation

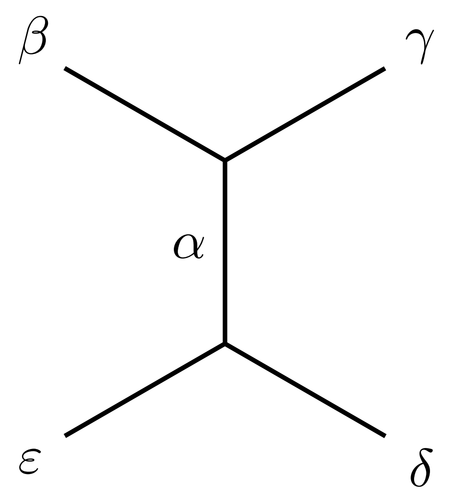
  edge $e \in E$ ⇨ generalised shear coordinate $z^e = x^e + \ell y^e \in R_\Lambda$

  face $f \in F$ ⇨ constraint $c^f_\Lambda = \sum_{\alpha \in f} \theta^f_\alpha z^\alpha$ $\quad \theta^f_\alpha \in \{1,2\}$ - multiplicity of α in f

- **moduli space of 3d gravity**

  moment map $c_\Lambda = (c^1_\Lambda, ..., c^F_\Lambda) : R^E_\Lambda \to R^F_\Lambda$

  moduli space of 3d gravity $\mathcal{M}_\Lambda(S) \cong \ker(c_\Lambda) \subset R^E_\Lambda$



- **symplectic structure in generalised shear coordinates**

  $\{x^\alpha, x^{\alpha'}\} = \{y^\alpha, y^{\alpha'}\} = 0 \qquad \{x^\alpha, y^{\alpha'}\} = \delta^{\beta\alpha'} + \delta^{\delta\alpha'} - \delta^{\gamma\alpha'} - \delta^{\epsilon\alpha'}$

  constraints Poisson-commute with generalised shear coordinates $\{c^f_\Lambda, z^\alpha\} = 0$

Theorem: [Scarinci, C.M.]

- induces a symplectic structure on $\mathcal{M}_\Lambda(S) = \mathrm{Hom}_F(\pi_1(S), G_\Lambda)/G_\Lambda \cong \ker(c_\Lambda) \subset R^E_\Lambda$ that coincides with Goldman's symplectic structure for $\kappa = \mathrm{Im}_\ell(K)$.
- $\pi^\sharp_{WP} : \mathbb{R}^E \times (\mathbb{R}^E)^* \to \mathbb{R}^E \times \mathbb{R}^E$, $(x^\alpha, p_\alpha) \mapsto (x^\alpha, \Sigma_{\beta \in E(\Gamma)} \pi^{\alpha\beta}_{WP} p_\beta)$ induces a symplectomorphism $\pi^\sharp_{WP} : T^*\mathcal{T}(S) \to \mathcal{M}_\Lambda(S)$.

Poisson structure and spacetime geometry - Teichmüller picture

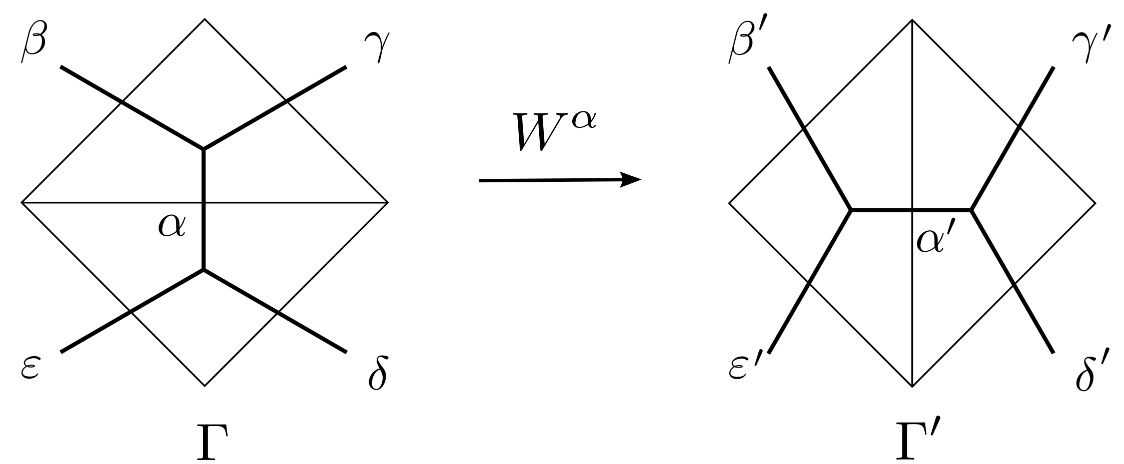
- **action of the mapping class group**

mapping class group $\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$

acts on generalised shear coordinates via Whitehead moves

induces $\mathrm{Mod}(S)$-action on $\mathcal{M}_\Lambda(S)$

$$W^\alpha : \begin{cases} z^\alpha \mapsto -z^\alpha \\ z^{\beta,\delta} \mapsto z^{\beta,\delta} + \log(1+e^{z^\alpha}) \\ z^{\gamma,\epsilon} \mapsto z^{\gamma,\epsilon} - \log(1+e^{-z^\alpha}) \end{cases}$$



Theorem: [Scarinci, C.M.]

The maps $W^\alpha : R^E_\Lambda \to R^E_\Lambda$ induce a $\mathrm{Mod}(S)$ - action on $\mathcal{M}_\Lambda(S)$ by symplectomorphisms.

- **geometrical interpretation** decompose Whitehead move $W^\alpha : R^E_\Lambda \to R^E_\Lambda$ as $W^\alpha = A^\alpha \circ B^\alpha$

linear part $A^\alpha : R^E_\Lambda \to R^E_\Lambda$

$$\begin{cases} z^\alpha \mapsto -z^\alpha \\ z^{\beta,\gamma,\delta,\epsilon} \mapsto z^{\beta,\gamma,\delta,\epsilon} + \frac{1}{2} z^\alpha \end{cases}$$

⇨ sum of shear coordinates along paths in Γ preserved

Hamiltonian part $B^\alpha : R^E_\Lambda \to R^E_\Lambda$

$$\begin{cases} z^\alpha \mapsto -z^\alpha \\ z^{\beta,\gamma,\delta,\epsilon} \mapsto z^{\beta,\gamma,\delta,\epsilon} + \{z^{\beta,\gamma,\delta,\epsilon}, \mathrm{Im}_\ell H_\alpha\} \end{cases}$$

⇨ ensures that holonomies along paths in Γ are preserved

Hamiltonian

$$\mathrm{Im}_\ell H_\alpha = \mathrm{Im}_\ell\left(\tfrac{1}{4}(z^\alpha)^2 + \mathrm{Li}_2(-e^{z_\alpha})\right) = \tfrac{1}{2}x^\alpha y^\alpha + \begin{cases} -y^\alpha \log(1+e^{x^\alpha}) & \ell^2 = 0 \\ \tfrac{1}{2}\mathrm{Li}_2(-e^{x^\alpha + y^\alpha}) - \tfrac{1}{2}\mathrm{Li}_2(-e^{x^\alpha - y^\alpha}) & \ell^2 = -1 \\ \mathrm{Im}\left(\mathrm{Li}_2(-e^{x^\alpha + \mathrm{i}y^\alpha})\right) & \ell^2 = -1 \end{cases}$$

~ related to volume of ideal tetrahedra