Lecture series on 3d gravity

Lecture 1:
Geometry of Classical 3d Gravity

Quantum Structure of Spacetime and Gravity 2016

August 21-28 2016
Belgrade, Serbia

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EUROPEAN COOPERATION
IN SCIENCE AND TECHNOLOGY

Motivation

Why 3d gravity?

- toy model for quantum gravity in higher dimensions
- non-commutative structures and mathematical structures of NC geometry: Hopf algebras, (co)module algebras, twists,…
- relates them to classical geometry: Lorentz and hyperbolic geometry, Teichmüller theory,…

Content

- **Lecture 1: Geometry of classical 3d gravity**
  - Construction of spacetimes
  - Classification results
  - Relation to Teichmüller and hyperbolic geometry
- **Lecture 2: Phase space and symplectic structure**
  - Phase space of 3d gravity
  - Symplectic structure in terms of Poisson-Lie groups
  - Phase space as cotangent bundle of Teichmüller space
- **Lecture 3: Quantisation**
  - Quantum 3d gravity as a Hopf algebra gauge theory
  - Construction of the quantum theory
  - Relation to models from condensed matter physics

1. Gravity in 3 dimensions

Einstein equations

$$\mathrm{Ric}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

- **in 3d:** Ricci curvature $\mathrm{Ric}_{\mu\nu}$ ➩ Riemann curvature $\mathrm{R}_{\mu\nu\rho\sigma}$
- no **local** gravitational degrees of freedom
- spacetimes locally isometric to **model spacetimes** X_Λ
- **global** degrees of freedom from matter (point particles) and topology
- **vacuum solutions** ($T_{\mu\nu} = 0$) ➩ constant curvature Λ ➩ **Einstein spacetimes**
- constructed as quotients of model spacetimes

model spacetimes

Lorentzian
$\mathrm{PSL}(2,\mathbb{R}) \subset G_\Lambda$
$X_\Lambda = G_\Lambda/\mathrm{PSL}(2,\mathbb{R})$

Euclidean
$\mathrm{SU}(2) \subset G_\Lambda$
$X_\Lambda = G_\Lambda/\mathrm{SU}(2)$

		spacetime X_Λ	isometry group G_Λ	
Lorentzian	$\Lambda > 0$	dS_3	$\mathrm{PSL}(2,\mathbb{C})$	$(-1,1,1)$
	$\Lambda = 0$	M_3	$\mathrm{Iso}(2,1) \cong \mathrm{PSL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$	
	$\Lambda < 0$	AdS_3	$\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$	
Euclidean	$\Lambda > 0$	S^3	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$(1,1,1)$
	$\Lambda = 0$	E_3	$\mathrm{Iso}(3) \cong \mathrm{SU}(2) \ltimes \mathbb{R}^3$	
	$\Lambda < 0$	$\mathbb{H}^3$	$\mathrm{PSL}(2,\mathbb{C})$	

2. Unified description of model spacetimes

isometry groups of Lorentzian model spacetimes

- **commutative real algebra** $R_\Lambda = (\mathbb{R}^2, +, \cdot_\Lambda)$:

$$(a,b)\cdot_\Lambda (c,d) = (ac - \Lambda bd, ad + bc)$$

notation: $(a,b) = a + \ell b$ with $\ell^2 = -\Lambda$

$$a = \mathrm{Re}_\ell(a+\ell b)$$
$$b = \mathrm{Im}_\ell(a+\ell b)$$
$$\overline{a+\ell b} = a - \ell b$$

- **analytic continuation** analytic function $f:\mathbb{R}\to\mathbb{R}$ ➩ analytic function $f: R_\Lambda \to R_\Lambda$

$$f(x+\ell y) = \begin{cases} f(x) + \ell f'(x) y & \Lambda = 0 \\ \tfrac{1}{2}(1+\ell) f(x+y) + \tfrac{1}{2}(1-\ell) f(x-y) & \Lambda = -1 \\ f(x+\mathrm{i}y) & \Lambda = 1 \end{cases}$$

➩ $$\frac{\partial \mathrm{Re}_\ell f}{\partial x} = \frac{\partial \mathrm{Im}_\ell f}{\partial y} \qquad \frac{\partial \mathrm{Re}_\ell f}{\partial y} = -\Lambda \frac{\partial \mathrm{Im}_\ell f}{\partial x}$$

- **isometry groups of model spacetimes**

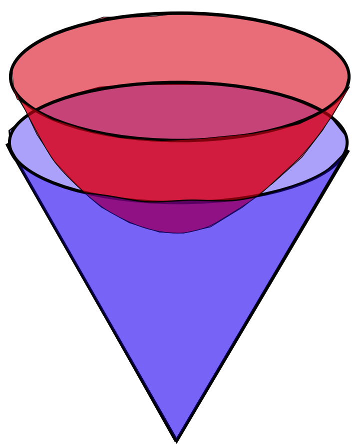
$$G_\Lambda = \{M \in \mathrm{Mat}(2, R_\Lambda) : \det(M) = 1\} = \begin{cases} \mathrm{Iso}(2,1) & \Lambda = 0 \\ \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) & \Lambda < 0 \\ \mathrm{SL}(2,\mathbb{C}) & \Lambda > 0 \end{cases}$$

- **Lie algebras of isometry groups**

$$\mathfrak{g}_\Lambda = \{M \in \mathrm{Mat}(2, R_\Lambda) : \mathrm{tr}(M) = 0\} = \begin{cases} \mathfrak{iso}(2,1) & \Lambda = 0 \\ \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) & \Lambda < 0 \\ \mathfrak{sl}(2,\mathbb{C}) & \Lambda > 0 \end{cases}$$

Lorentzian model spacetimes: M_3, dS_3, AdS_3

- **involution** $\circ : \mathrm{Mat}(2, R_\Lambda) \to \mathrm{Mat}(2, R_\Lambda)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar d & -\bar b \\ -\bar c & \bar a \end{pmatrix}$
- **model spacetimes** $X_\Lambda = \{M \in \mathrm{Mat}(2, R_\Lambda) : M^\circ = M, \det(M) = 1\}$ for $\Lambda \in \{0, \pm 1\}$
- **action of isometry group** $\rhd : G_\Lambda \times X_\Lambda \to X_\Lambda \qquad G \rhd M = G \cdot M \cdot G^\circ$
- **metric** $\langle M, M\rangle = -\det(\mathrm{Im}_\ell(M) + \ell\, \mathrm{Re}_\ell(M))$
- **geodesics** $g(t) = M \exp(t\ell\, X)$ for $M \in X_\Lambda$, $X \in \mathfrak{sl}(2, \mathbb{R})$
- **standard future lightcone** $L = \{\exp(\ell\, X) : X \in \mathfrak{sl}(2, \mathbb{R}), \mathrm{tr}(X^2) < 0\}$



- **foliation of lightcone by 2d hyperbolic space**

$$H : \mathbb{R} \times \mathbb{H}^2 \to X_\Lambda \qquad H(t, z) = c_\Lambda(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\ell s_\Lambda(t)}{\mathrm{Im}(z)} \begin{pmatrix} -\mathrm{Re}(z) & |z|^2 \\ -1 & \mathrm{Re}(z) \end{pmatrix}$$

$$c_\Lambda(t) = \sum_{k=0}^{\infty} \frac{t^{2k}\Lambda^k}{(2k)!} = \begin{cases} 1 & \Lambda = 0 \\ \cos(t) & \Lambda = -1 \\ \cosh(t) & \Lambda = 1 \end{cases} \qquad s_\Lambda(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1}\Lambda^k}{(2k+1)!} = \begin{cases} t & \Lambda = 0 \\ \sin(t) & \Lambda = -1 \\ \sinh(t) & \Lambda = 1 \end{cases}$$

⇨ compatible with $\mathrm{SL}(2, \mathbb{R})$ -action: $H(g \rhd z, t) = g \cdot H(z, t) \cdot g^\circ$

↑ action on upper half-plane by Möbius transformations

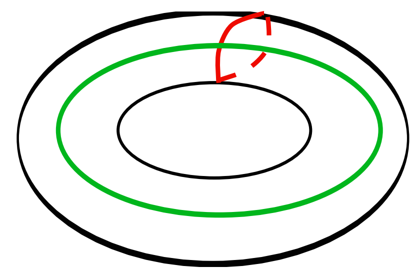
3. Construction and classification of spacetimes

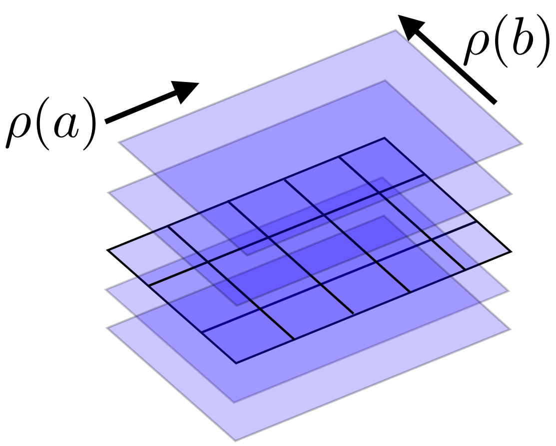
maximal globally hyperbolic Lorentzian spacetimes M with compact Cauchy surface S

- ➯ M homeomorphic to $\mathbb{R} \times S$
- ➯ universal cover $\tilde{M} \subset X_\Lambda$ globally hyperbolic
- ➯ $M = \tilde{M}/\pi_1(S)$, $\pi_1(S) \curvearrowright \tilde{M}$ via group homomorphism $\rho : \pi_1(S) \to G_\Lambda$

universal cover

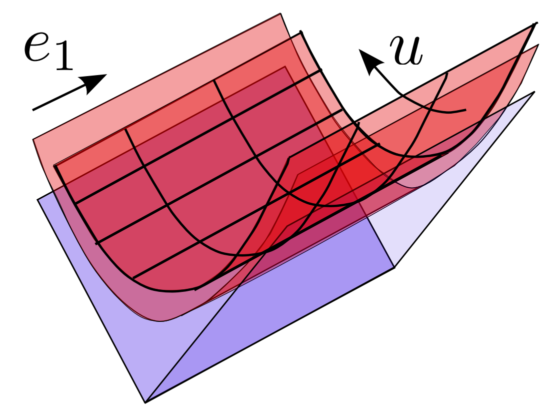
- **Ex: torus universe for Λ=0** $\quad \pi_1(S) = \mathbb{Z} \times \mathbb{Z}$





$\rho(a), \rho(b) \in \mathbb{R}^3$
spacelike translations

$\tilde{M} = M_3$



$\rho(a) = e_1 \in \mathbb{R}^3$
spacelike translation

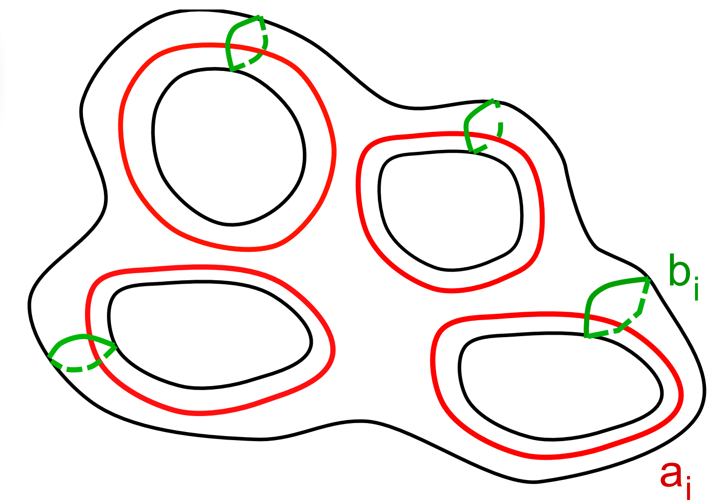
$\rho(b) = u \in \mathrm{SO}(2,1)$
Lorentz boost
stabilising e_1

$\tilde{M} = I_+(\mathbb{R}e_1)$
future of a line

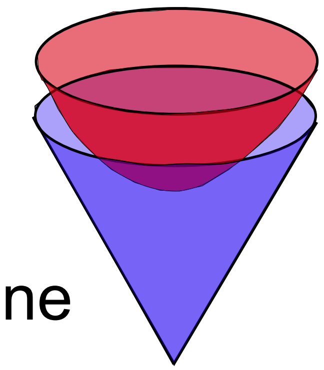
- **Cauchy surface of genus g>1, general Λ** $\quad$ [Mess, Benedetti, Bonsante]
  - $\tilde{M}$ is convex, open, future complete region in X_Λ , future of a spacelike graph
  - initial singularity $\partial\tilde{M}$
  - cosmological time function
    $t : \tilde{M} \to \mathbb{R} \qquad t(p) = \sup\{l(c) : c \text{ past directed causal curve with } c(0) = p\}$
  - foliation by surfaces of constant cosmological time (cct) $\quad \tilde{M} = \cup_T \tilde{M}_T \qquad \tilde{M}_T = t^{-1}(T)$
  - $\pi_1(S) \curvearrowright \tilde{M}_T$ and $M = \cup_T \tilde{M}_T/\pi_1(S)$

conformally static spacetimes of genus g>1

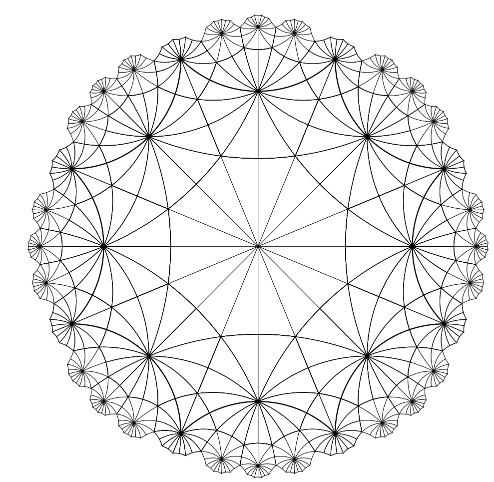
$$\pi_1(S) = \langle a_1, b_1, ..., a_g, b_g \,|\, [a_g, b_g] \cdots [a_1, b_1] = 1 \rangle$$



- **group homomorphism** $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$
- **universal cover** $\tilde{M} = L$ - interior of standard lightcone

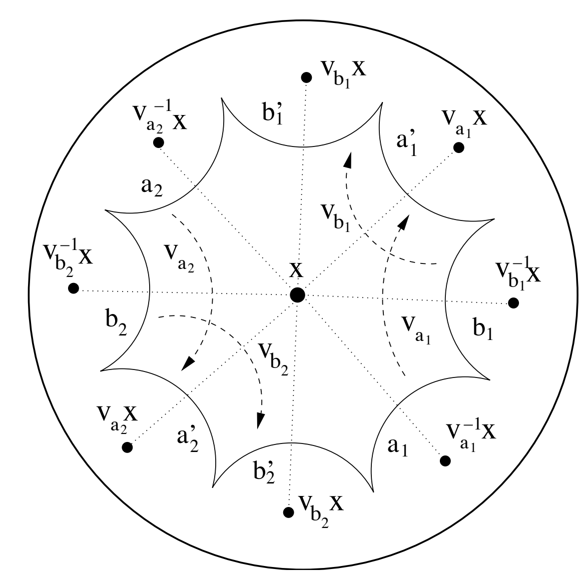


- **cosmological time** $t : \tilde{M} \to \mathbb{R}$ - geodesic distance from tip of lightcone
- **cct surfaces** $\tilde{M}_T = H(\mathbb{H}^2, T) \cong s_\Lambda(T)\, \mathbb{H}^2$ - rescaled copies of $\mathbb{H}^2$



- **action of $\pi_1(S)$**
  - ➪ action of Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}^2$
  - ➪ tesselation of $\mathbb{H}^2$ by geodesic arc 4g-gons
  - ➪ $\mathrm{PSL}(2, \mathbb{R})$ covariant: $\tilde{M} \to g \cdot \tilde{M} \cdot g^\circ$ corresponds to $\rho \to g \cdot \rho \cdot g^\circ$
- **spacetime** $M = \cup_T \tilde{M}_T / \pi_1(S) = \cup_T\, s_\Lambda(T)\, \mathbb{H}^2 / \Gamma$ conformally static

$$g_M = -dT^2 + s_\Lambda(T)^2 g_{\mathbb{H}^2/\Gamma}$$



for all values of Λ:

$$\{\text{conformally static MGH spacetimes of genus } g > 1\}/\mathrm{Diff}_0(M)$$
$$= \{\text{Fuchsian groups } \Gamma \subset \mathrm{PSL}(2, \mathbb{R}) \text{ of genus } g > 1\}/\mathrm{PSL}(2, \mathbb{R})$$
$$= \mathcal{T}(S)$$ Teichmüller space

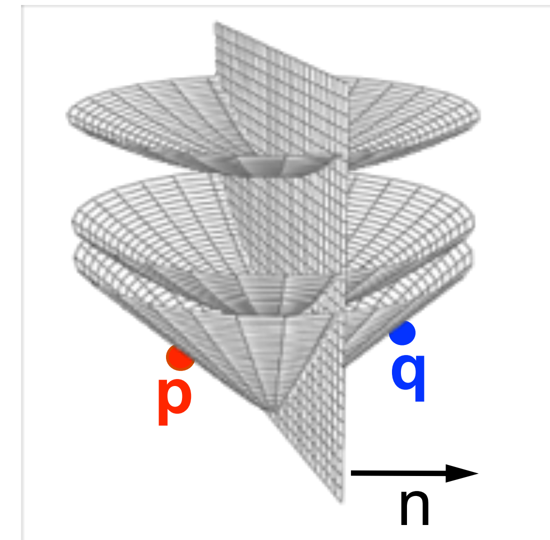
geometry change via earthquake and grafting

ingredients

- hyperbolic surface $\Sigma = \mathbb{H}^2/\Gamma$ ⇔ cocompact Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$
  ⇔ conformally static spacetime
- weighted multicurve $\{(c_i, w_i)\}_{i\in I}$ finite set of closed, non-intersecting geodesics c_i on Σ
  with weights $w_i > 0$

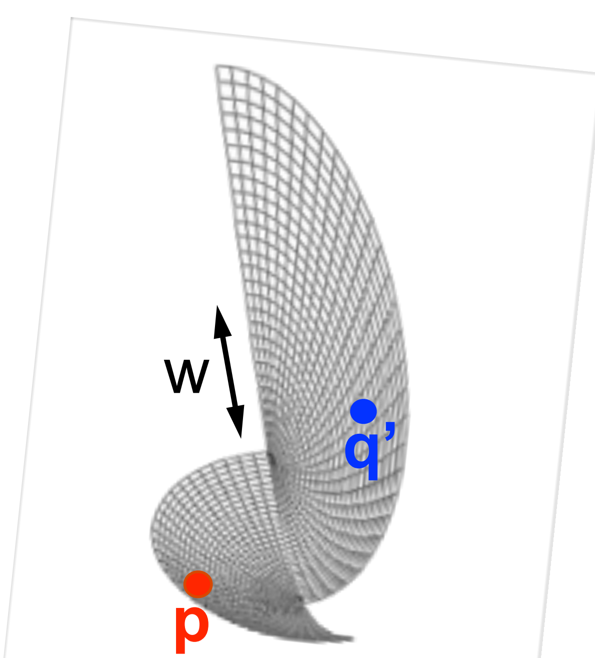
construction

- lift geodesics to $\mathbb{H}^2$ and embed into foliated lightcone
- select basepoint
- cut lightcone along geodesics



earthquake

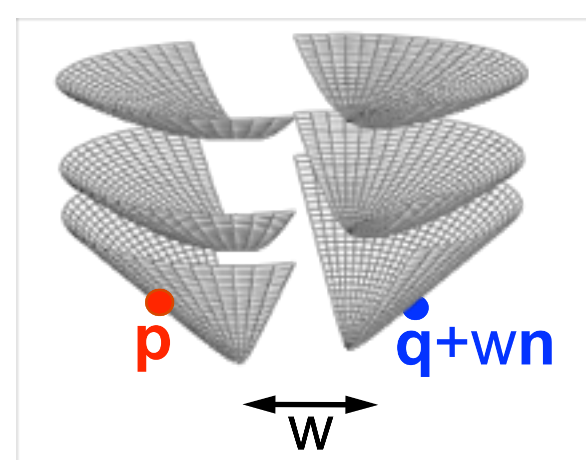
- for each geodesic c_i :
  apply $\exp(w_i X_{c_i}) \in \mathrm{PSL}(2,\mathbb{R})$ to the right
  Lorentz boost, hyperbolic distance w_i

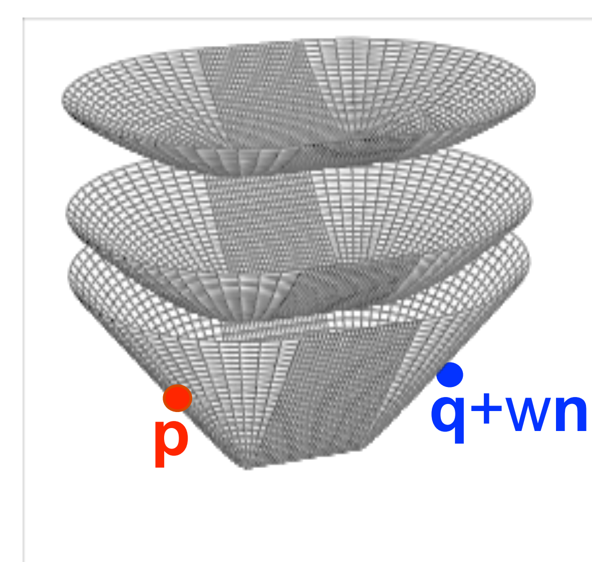


$X_{c_i} \in \mathfrak{sl}(2,\mathbb{R})$
$\exp(sX_{c_i}) \in \mathrm{PSL}(2,\mathbb{R})$ stabilises c_i
$\mathrm{tr}(X_{c_i}^2) = \frac{1}{2}$
moves to the left

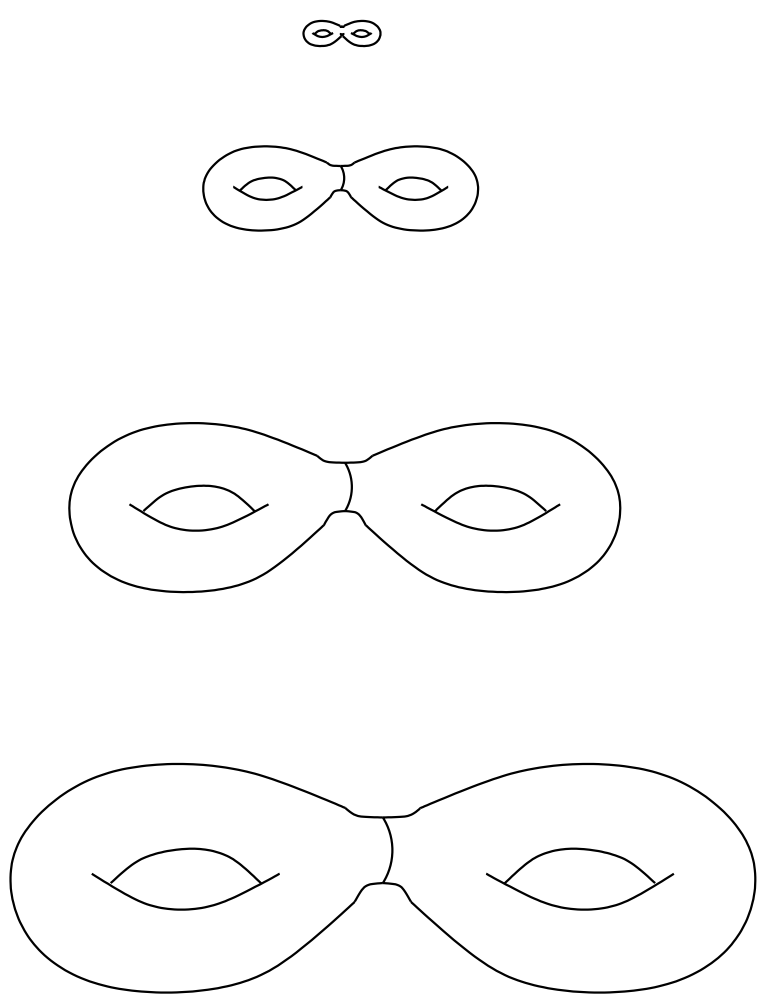
grafting

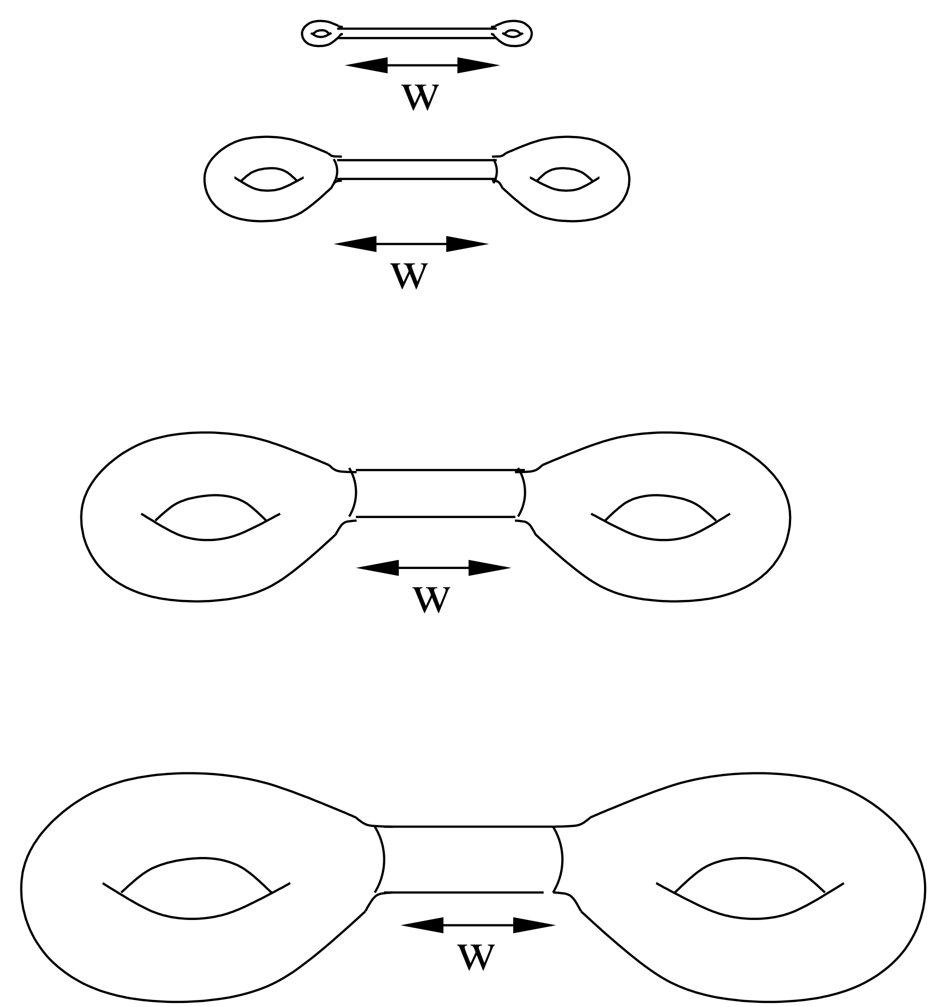
- for each geodesic c_i :
  apply $\exp(\ell\, w_i X_{c_i}) \in G_\Lambda$ to the right
  ~ imaginary earthquake
  translation, distance w_i





	earthquake	grafting
• **universal cover**	• remains standard lightcone • future of a point	• deformed lightcone • future of a graph
• **cosmological time**	• geodesic distance from tip of lightcone	• geodesic distance from graph
• **ccT surfaces**	• rescaled copies of $\mathbb{H}^2$	• deformed copies of $\mathbb{H}^2$
• **action of $\pi_1(S)$**	• via group homomorphism $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$	• via group homomorphism $\rho : \pi_1(S) \to G_\Lambda$
• **spacetime**	• remains conformally static	• evolves with cosmological time

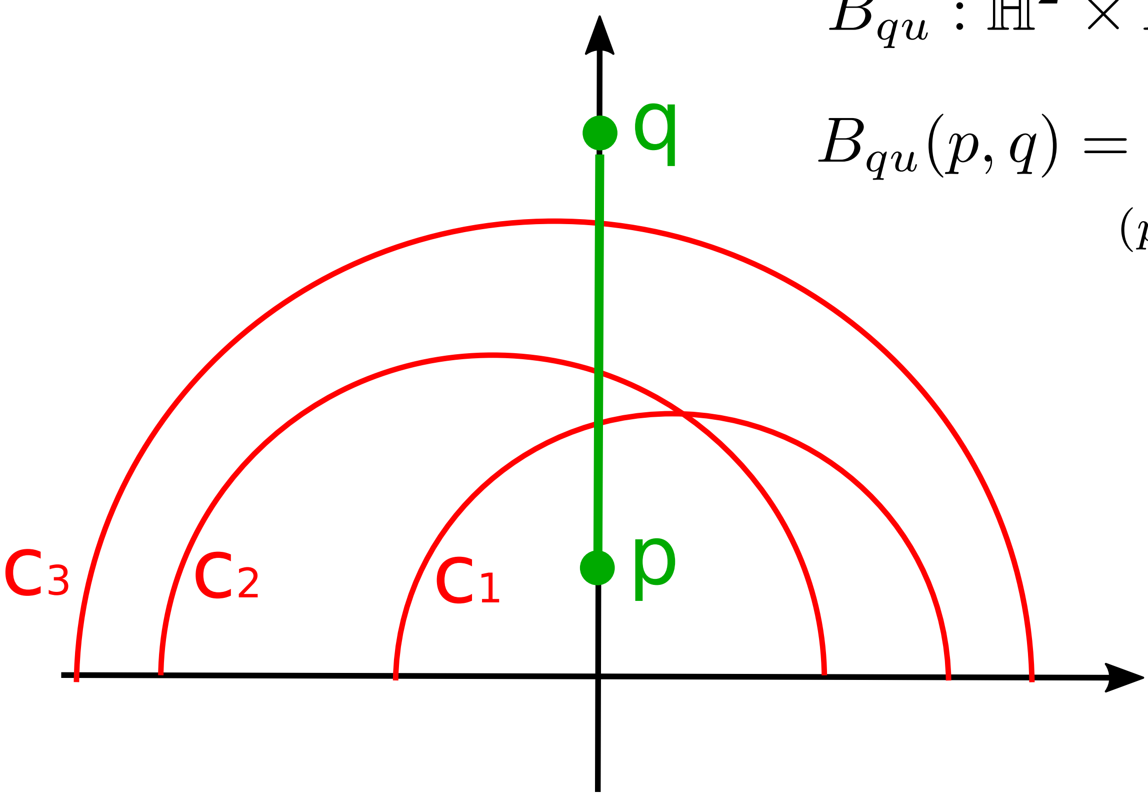




earthquake and grafting - transformation of holonomies

- group homomorphism $\rho_0 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ ⇔ Fuchsian group $\Gamma = \mathrm{im}(\rho_0)$ of genus g
- weighted multicurve $\{(c_i, w_i)\}_{i\in I}$ on $\Sigma = \mathbb{H}^2/\Gamma$

transformation of group homomorphism given by cocycles



earthquake

$$B_{qu} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathrm{PSL}(2,\mathbb{R})$$

$$B_{qu}(p,q) = \prod_{(p,q)\cap c_i} \exp(w_i X_{c_i})$$

grafting

$$B_{gr} : \mathbb{H}^2 \times \mathbb{H}^2 \to G_\Lambda$$

$$B_{gr}(p,q) = \prod_{(p,q)\cap c_i} \exp(\ell w_i X_{c_i})$$

~ imaginary earthquake

$X_{c_i} \in \mathfrak{sl}(2,\mathbb{R})$

$\exp(sX_{c_i}) \in \mathrm{PSL}(2,\mathbb{R})$ stabilises c_i

$\mathrm{tr}(X_{c_i}^2) = \frac{1}{2}$

moves to the left

- $\mathrm{PSL}(2,\mathbb{R})$- covariant $\quad B(g \rhd p, g \rhd q) = g \cdot B(p,q) \cdot g^\circ \quad g \in \mathrm{PSL}(2,\mathbb{R})$
- Γ- invariant $\quad B(g \rhd p, g \rhd q) = B(p,q) \quad g \in \Gamma$

transformation of holonomies $\rho : \pi_1(S) \to G_\Lambda$

$$\rho(\lambda) = \rho_0(\lambda) \cdot B(p, \rho_0(\lambda) \rhd p)$$

classification results

[Thurston, Mess, Barbot, Benedetti, Bonsante, Schlenker],…

- **characterisation of MGH spacetimes by earthquake and grafting**

⇨ consider generalisation of earthquake and grafting to measured geodesic laminations (= limits of weighted multicurves with infinitely many geodesics)

Theorem: [Thurston, Mess,Bendetti-Bonsante, Schlenker]

S compact of genus g>1:

- every evolving spacetime is obtained from conformally static spacetime via grafting along a measured geodesic lamination
- every conformally static spacetime is obtained from given conformally static spacetime via earthquake along a measured geodesic lamination

- **characterisation of MGH spacetimes by group homomorphisms**

Theorem: [Mess,Bendetti-Bonsante]

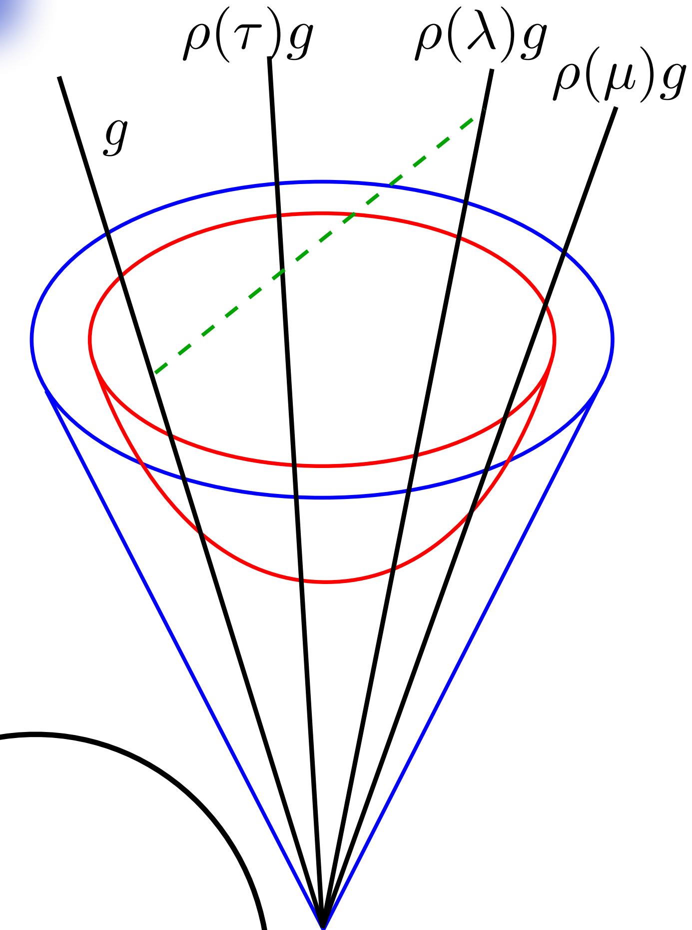
S compact of genus g>1:

- group homomorphisms related by conjugation determine isometric spacetimes
- $\Lambda \leq 0: \ \rho : \pi_1(S) \to G_\Lambda$ determines M up to diffeomorphisms
- $\Lambda > 0: \rho : \pi_1(S) \to G_\Lambda$ determines M up to diffeomorphisms & up to discrete parameter

- similar results for Cauchy surfaces S with cusps or punctures (⇨ weaker)

physics: measuring the group homomorphisms

[C.M.]



example: conformally static spacetime of genus g>1, Λ=0

determined by $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$

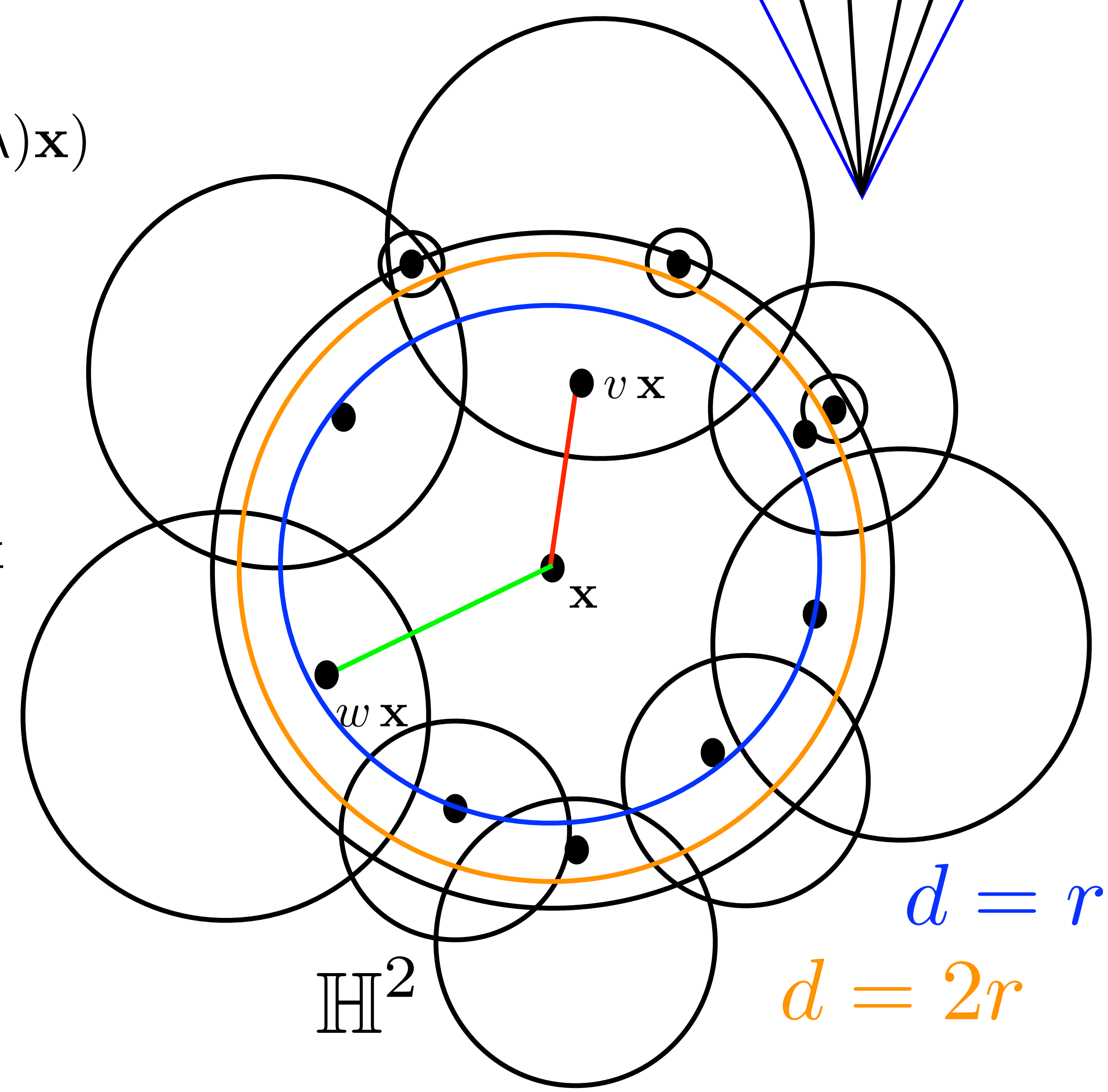
- **observer** $\pi_1(S)$ - equivalence class of timelike, future directed geodesic in $\tilde{M}$

- **returning light signal** lightlike, future directed geodesic from one observer geodesic g to an image $\rho(\lambda)g$

- **return time** $\Delta t = \langle \dot{g}(t), \rho(\lambda)\dot{g}(t)\rangle$

  $g(t) = \mathbf{x}t$ $\qquad = \langle \mathbf{x}, \rho(\lambda)\mathbf{x}\rangle = \cosh d_{\mathbb{H}^2}(\mathbf{x}, \rho(\lambda)\mathbf{x})$

- **measurement of group homomorphism**
- observer emits light in all directions
- measures return time and direction of signals
- draws geodesic segment through $\mathbf{x}$ and $\rho(\lambda)\mathbf{x}$
- constructs perpendicular bisector



⇨ **observer reconstructs Dirichlet region of $\rho(\pi_1(S)) \subset \mathrm{PSL}(2,\mathbb{R})$ and $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ up to conjugation in finite eigentime**

classification in terms of moduli spaces

conclusion: $M = \mathbb{R} \times S,\ S$ compact of genus $g > 1$

phase space of 3d gravity contained in **moduli space of flat G_Λ-connections on S**

$$\mathcal{M}_\Lambda(S) = \{\text{max. glob. hyperbolic Lorentzian structures on } M \text{ of curvature } \Lambda\}/\text{Diff}_0(M)$$

$$\text{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$$

similarly: S compact of genus $g > 1$

Teichmüller space contained in **moduli space of flat $\text{PSL}(2,\mathbb{R})$-connections on S**

$$\mathcal{T}(S) = \{\text{hyperbolic structures on } S\}/\text{Diff}_0(S)$$

$$\text{Hom}(\pi_1(S), \text{PSL}(2,\mathbb{R}))/\text{PSL}(2,\mathbb{R})$$

- **not coincidental:**
  - ➪ spacetimes obtained by grafting along measured geodesic laminations
  - ➪ measured geodesic laminations form fibre bundle $\mathcal{ML}(\mathcal{S})$ over Teichmüller space $\mathcal{T}(S)$

$$\mathcal{M}_\Lambda(S) \cong \mathcal{ML}(\mathcal{S}) \cong T^*\mathcal{T}(S)$$

➪ description of phase space $\mathcal{M}_\Lambda(S)$ in terms of structures from Teichmüller theory

➪ for surfaces with cusps: simple description in terms of shear coordinates

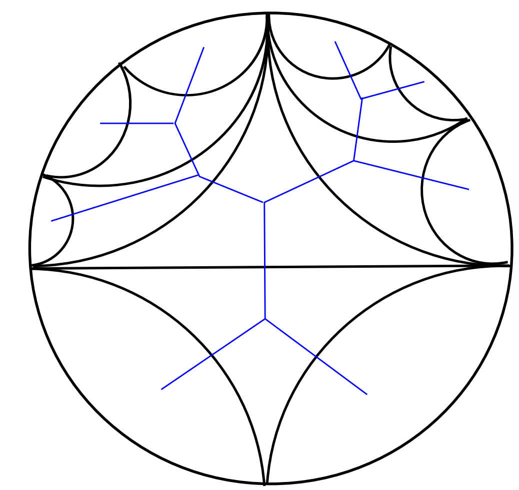
4. 3d spacetimes from Teichmüller space

S oriented surface of genus g with s>0 punctures (cusps), 2g-2+s>0

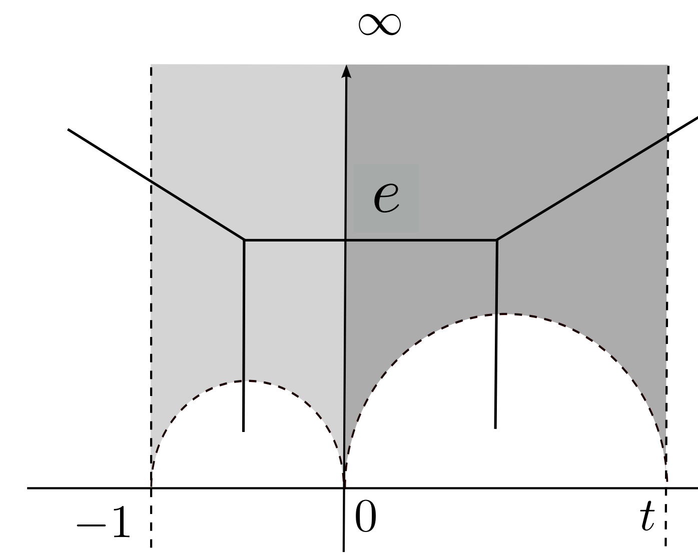
- **Teichmüller space**

$$\mathcal{T}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}_0(S) = \mathrm{Hom}_F(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

↑ finite area hyperbolic metrics on S with cusps at punctures



- **parametrisation by shear coordinates:**
  - ideal triangulation of S: edges geodesic segments, all vertices at cusps
    - ➪ dual graph Γ = trivalent graph
    - ➪ triangulation lifts to geodesic triangulation of $\mathbb{H}^2$, vertices at $\partial\mathbb{H}^2$
    - ➪ assignment of ideal square to edge $e \in E$

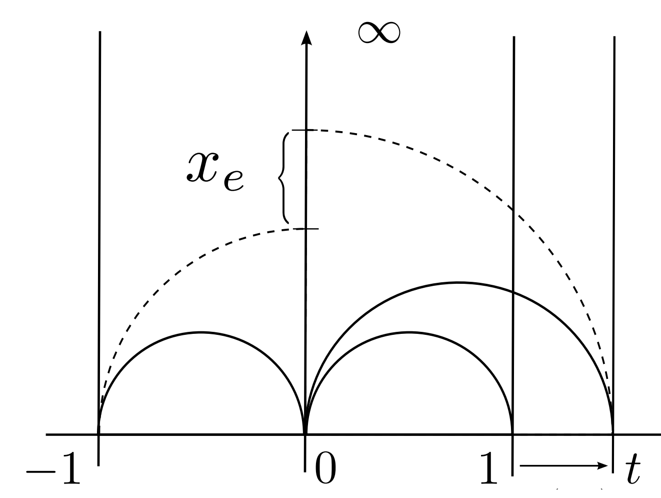


- shear coordinate for $e \in E$: $x_e(h) = \log t = \log \dfrac{(t-0)\cdot(-1-\infty)}{(-1-0)\cdot(t-\infty)}$

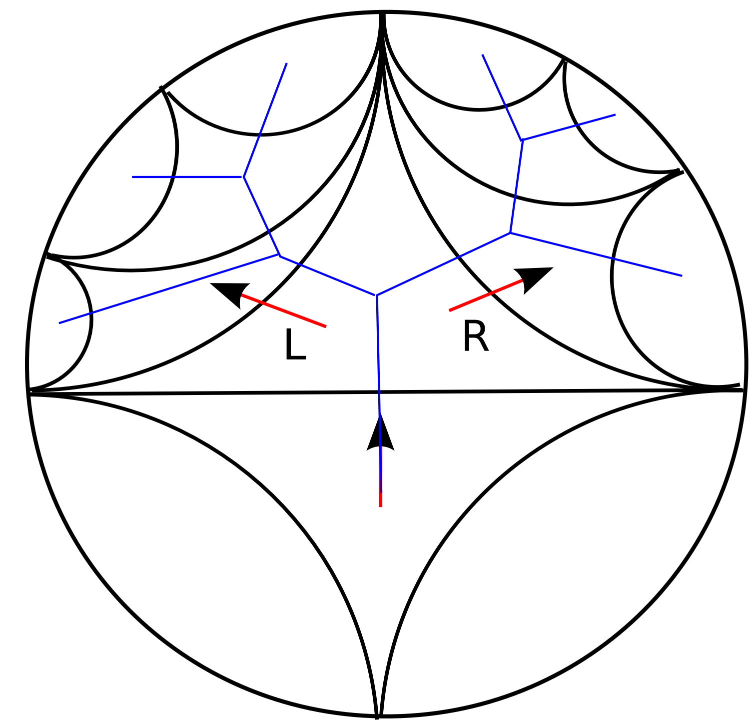
cross ratio of ideal square

- geometrical interpretation:
  - reference edge $(0,\infty)$, reference triangles $(-1,0,\infty)$, $(0,1,\infty)$
  - triangle $(0,t,\infty)$ from $(0,1,\infty)$ via earthquake along $(0,\infty)$ with weight $x_e(h) = \log t$

$$E(x) = \begin{pmatrix} e^{x/2} & 0 \\ 0 & e^{-x/2} \end{pmatrix} : z \mapsto e^x\, z$$



shear coordinates and holonomies



$\lambda \in \pi_1(S)$ ➩ edge sequence $\lambda = (\alpha_1, ..., \alpha_n)$ in Γ

➩ sequence of left /right turns at vertices of Γ

- **holonomies**

$$\lambda \mapsto \rho(\lambda) = P^a_n E(x^{\alpha_n}) P^a_{n-1} E(x^{\alpha_{n-1}}) \cdots P^a_1 E(x^{\alpha_1}) \in \mathrm{PSL}(2, \mathbb{R})$$

$$P^a_k = L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad P^a_k = R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

length of associated geodesic on S : $\mathrm{tr}(\rho(\lambda)) = 2\cosh(l(\lambda)/2)$

- **faces of Γ ➩ consistency conditions**

closed paths that turn right at each vertex and traverse edges at most once in each direction

➩ $\mathrm{tr}(\rho(f)) = 2\cosh(c^f)$ with $c^f = \sum_{\alpha \in f} \theta^f_\alpha x^\alpha$ $\theta^f_\alpha \in \{1, 2\}$ - multiplicity of α in f

faces ~ loops around cusps ➩ holonomies must be parabolic $\mathrm{tr}(\rho(f)) \stackrel{!}{=} 2 \Leftrightarrow c^f \stackrel{!}{=} 0$

➩ moment map $c = (c^1, ..., c^F) : \mathbb{R}^E \to \mathbb{R}^F$

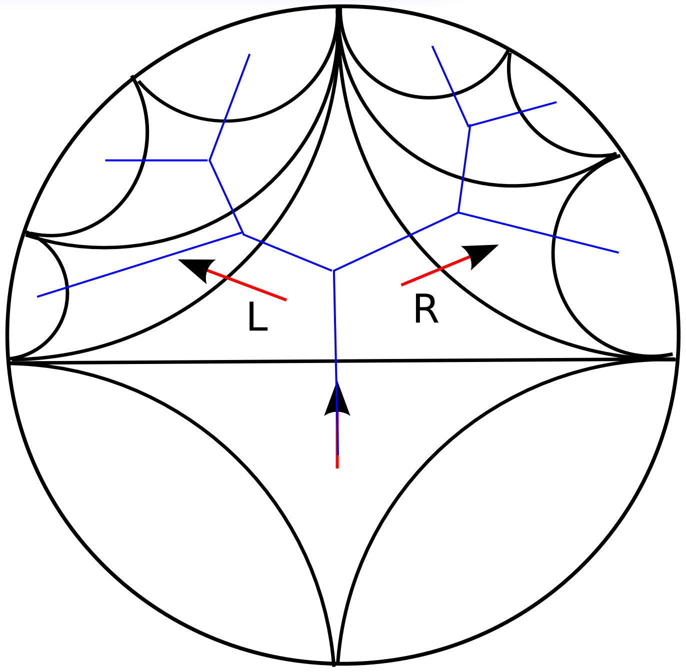
Theorem: [Fock-Checkov, Penner]

Teichmüller space

$$\mathcal{T}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}_0(S) = \mathrm{Hom}_F(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R}) \cong \ker(c)$$

moduli spaces of 3d gravity by analytic continuation of shear coordinates

- trivalent graph Γ dual to ideal triangulation



- **analytic continuation of shear coordinates**

  assign to edge $e \in E$ shear coordinate $z^e = x^e + \ell y^e \in R_\Lambda$

⇨ consider $R_\Lambda^E = (x^1 + \ell y^1, ..., x^E + \ell y^E)$

- **faces of Γ ⇨ moment maps**

  $c_\Lambda = (c_\Lambda^1, ..., c_\Lambda^F) : R_\Lambda^E \to R_\Lambda^F$

  $c_\Lambda^f(x^1 + \ell y^1, ..., x^E + \ell y^E) = \sum_{\alpha \in f} \theta_\alpha^f (x^\alpha + \ell y^\alpha)$ $\quad \theta_\alpha^f \in \{1, 2\}$ - multiplicity of α in f

Theorem [Scarinci, C.M.]

- moduli spaces of 3d gravity $\mathcal{M}_\Lambda(S) \cong \ker(c_\Lambda) \subset R_\Lambda^E$
- group homomorphisms $\rho : \pi_1(S) \to G_\Lambda$

  for closed path $\lambda = (\alpha_1, ..., \alpha_n)$ in Γ

  $\rho(\lambda) = P_n^a E(z^{\alpha_n}) \cdots P_1^a E(z_1^\alpha)$ with $P_k^a = L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $P_k^a = R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$E(x + \ell y) = \begin{pmatrix} e^{\frac{1}{2}(x+\ell y)} & 0 \\ 0 & e^{-\frac{1}{2}(x+\ell y)} \end{pmatrix} = \begin{cases} (E(x), ye_1) & \Lambda = 0 \\ (E(x+y), E(x-y)) & \Lambda = -1 \\ E(x + iy) & \Lambda = 1 \end{cases}$$

5. The action of the mapping class group

mapping class group of oriented surface S

$$\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S) = \mathrm{Out}(\pi_1(S)) = \mathrm{Aut}(\pi_1(S))/\mathrm{Inn}(\pi_1(S))$$

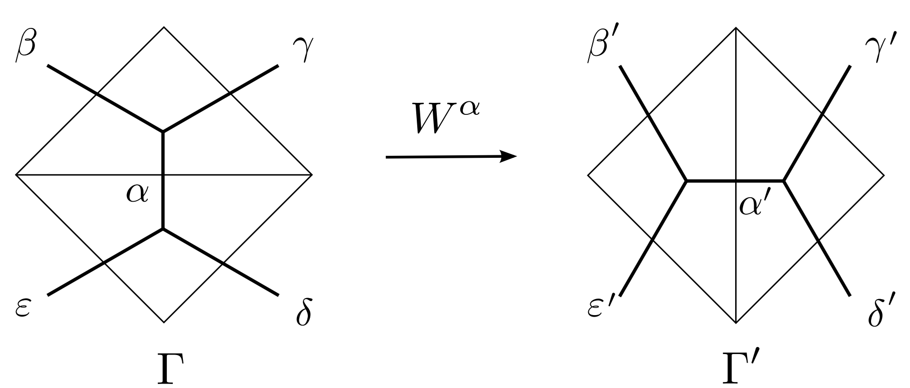
- action on $\mathrm{Hom}(\pi_1(S), G)/G$ ⇨ essential in quantisation

$$\phi \in \mathrm{Aut}(\pi_1(S)):\ \rho : \pi_1(S) \to G \quad \to \quad \rho \lhd \phi = \rho \circ \phi : \pi_1(S) \to G$$

- simple description of in terms of triangulations

$\mathrm{Mod}(S)$ acts by:

- finite sequences of Whitehead moves



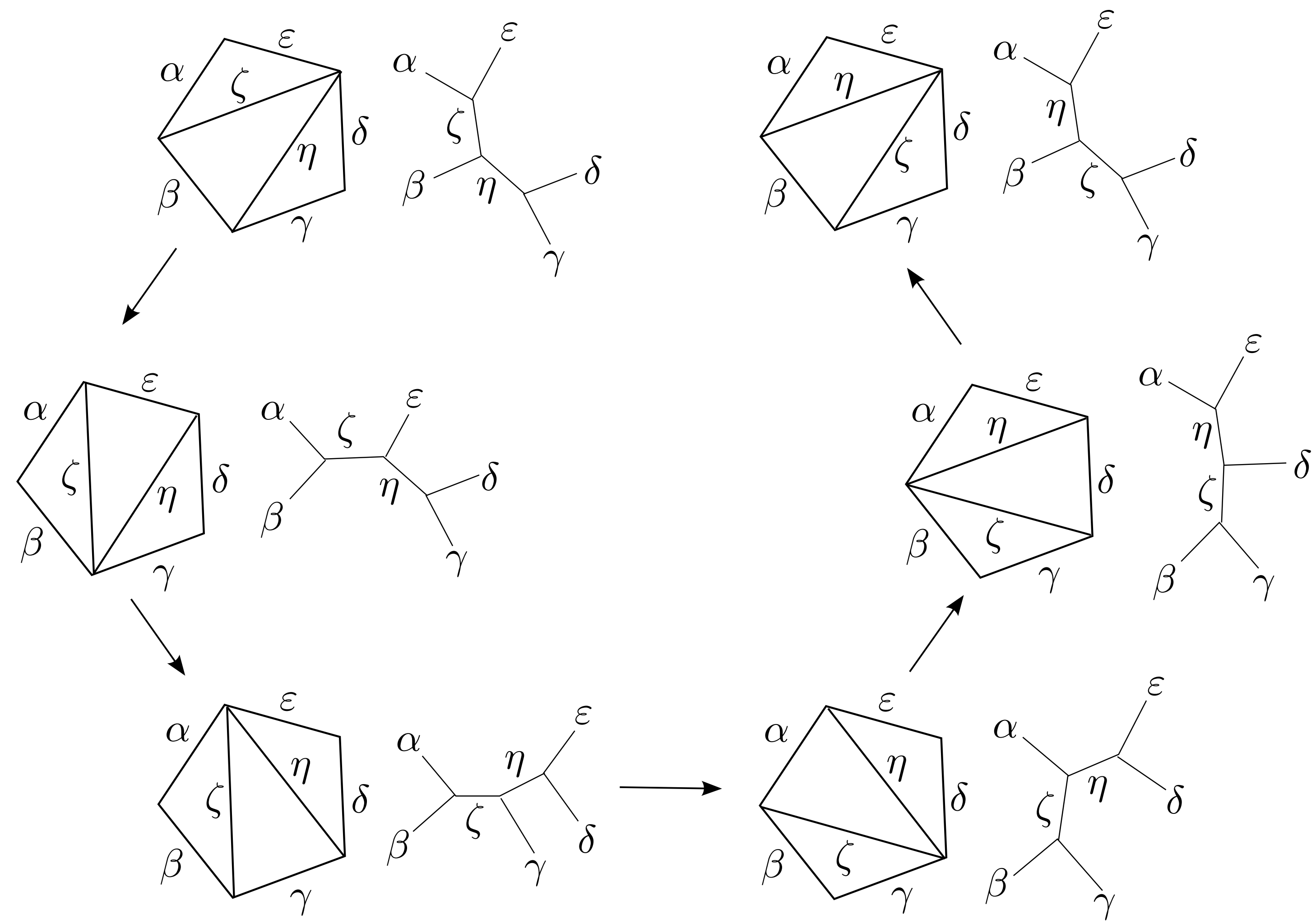
- modulo relations

$$(W^\alpha)^2 = \mathrm{id}$$

$$W^\alpha \circ W^\beta = W^\beta \circ W^\alpha \quad \text{for} \quad \alpha \cap \beta = \emptyset$$

$$(\alpha\beta) \circ W_\alpha = W_\beta$$

$$W^\zeta \circ W^\eta \circ W^\zeta \circ W^\eta \circ W^\zeta = (\zeta\eta)$$ pentagon



Mod(S) - action on Teichmüller space

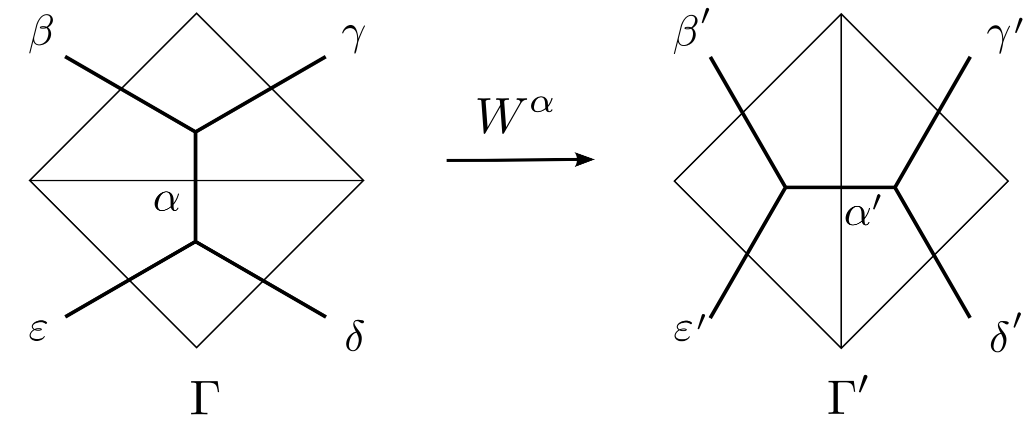
$$\mathcal{T}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}_0(S)$$

relation to Riemann moduli space

$$\mathrm{Riem}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}^+(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$$

transformation of shear coordinates under Whitehead move

$$W_\alpha : \begin{cases} x^\alpha \mapsto -x^\alpha \\ x^{\beta,\delta} \mapsto x^{\beta,\delta} + \log(1+e^{x^\alpha}) \\ x^{\gamma,\epsilon} \mapsto x^{\gamma,\epsilon} - \log(1+e^{-x^\alpha}) \end{cases}$$



Theorem: [Fock-Checkov, Penner]

- transformation of shear coordinates defines $\mathrm{Mod}(S)$ - action on $\mathbb{R}^E$
- preserves the constraints $c = c' \circ W_\alpha$ and induces a $\mathrm{Mod}(S)$ -action on $\mathcal{T}(S) \cong \ker(c) \subset \mathbb{R}^E$

Mod(S) - action on moduli spaces of 3d gravity

transformation of generalised shear coordinates under Whitehead move

$$W^\alpha : \begin{cases} z^\alpha \mapsto -z^\alpha \\ z^{\beta,\delta} \mapsto z^{\beta,\delta} + \log(1+e^{z^\alpha}) \\ z^{\gamma,\epsilon} \mapsto z^{\gamma,\epsilon} - \log(1+e^{-z^\alpha}) \end{cases}$$

$$z^e = x^e + \ell y^e \in R_\Lambda$$

Theorem: [Scarinci, C.M.]

The Whitehead moves $W^\alpha : R_\Lambda^E \to R_\Lambda^E$ satisfy the pentagon relation, preserve the constraints $c_\Lambda : R_\Lambda^E \to R_\Lambda^F$ and induces an action of $\mathrm{Mod}(S)$ on $\mathcal{M}_\Lambda(S)$

6. Summary

- **unified description of Lorentzian model spacetimes and isometry groups for different values of Λ**

- **unified description of MGH Lorentzian spacetimes as quotients of universal cover**
    - conformally static spacetimes:
      via action of Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ on lightcone
      related by earthquakes
    - evolving spacetimes:
      from conformally static spacetimes via grafting

- **diffeomorphism classes of MGH spacetimes**
  ⇔ conjugacy classes of group homomorphisms $\rho : \pi_1(S) \to G_\Lambda$

- **phase space of 3d gravity contained in moduli space of flat** G_Λ **-connections on** S

$$\mathcal{M}_\Lambda(S) = \{\text{max. glob. hyperbolic Lorentzian structures on } M \text{ of curvature } \Lambda\}/\mathrm{Diff}_0(M) \quad \subset \quad \mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$$

- **relation to Teichmüller space: via analytic continuation of shear coordinates**

- **explicit description of mapping class group action on** $\mathcal{M}_\Lambda(S)$